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Equations of Motion in Fourth Approximation

J. M. Gambi,[1] A. San Miguel,[2] and F. Vicente[2]

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The equations of motion in fourth approximation for a system of massive bodies of finite size moving in the gravitational field of the system are obtained.

1. INTRODUCTION

The purpose of this paper is to obtain equations of motion in fourth approximation for a system of massive bodies of finite size moving in its own gravitational field by means of Synge's approximation method (Synge, 1970).

In earlier papers, Synge's method has been applied in third approximation to the study of the motion of several systems of massive bodies and their associated fields (Hogan and McCrea, 1974; McCrea and O'Brien, 1978; O'Brien, 1979; Gambi, 1983, 1985; Gambi and San Miguel, 1986) and in fourth approximation to the study of the lowest order radiation terms in connection with the quadrupole formula (McCrea, 1981).

Synge's third approximation includes Chandrasekhar's (1965) first post-Newtonian approximation and goes part of the way from that to the second post-Newtonian approximation of Chandrasekhar and Nutku (1969). The post-Newtonian approximation (PNA) was first carried far enough to give radiation terms by Chandrasekhar and Esposito (1970) and subsequently put on a more systematic mathematical basis by Anderson and Decanio (1975).

The main difference between Synge's approach and that of Chandrasekhar and of Chandrasekhar and Nutku lies in the gauge conditions used. Whereas the conditions of these authors lead to Poisson equations and

[1]CEME Department, University of Vermont, Burlington, Vermont, USA.
[2]Department of Applied Mathematics, University of Valladolid, Valladolid, Spain.

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instantaneous potential solutions, Synge's conditions lead to inhomogeneous wave equations and retarded potential solutions as in the scheme of Anderson and Decanio. In this respect, Synge's method is closer to that of these authors, although their gauge conditions and corresponding energy pseudotensors are also different. Furthermore, the convergence of all the integrals appearing in Synge scheme is guaranteed from the beginning, including those which occur when the retarded potentials are expanded in terms of instantaneous potentials (Synge, 1970; McCrea, 1981).

Synge (1970) contains explicitly the equations of motion in third approximation, which have been applied to the study of the motions mentioned before. In this paper we develop Synge's scheme to one further stage, obtain the equations of motion in fourth approximation, and then apply them to a concrete model.

2. NOTATION AND GENERAL METHOD

For details of Synge's method the reader is referred to Synge (1970). For a given energy tensor T^{ab} we generate a sequence of metrics

$$\underset{m}{g}_{ab} = \delta_{ab} + \underset{m}{\gamma}_{ab}, \qquad (m = 0, 1, 2, \ldots, N) \tag{1}$$

by the recurrence formula

$$\underset{m}{\gamma}^*_{rs} = 2\kappa K^{rs}_{ab} \underset{m-1}{H}^{ab}, \qquad (m = 1, 2, \ldots, N) \tag{2}$$

where

$$\underset{m}{\gamma}^*_{ab} = \underset{m}{\gamma}_{ab} - \tfrac{1}{2}\delta_{ab} \underset{m}{\gamma}_{dd} \tag{3}$$

and

$$\underset{m}{H}^{ab} = T^{ab} + \kappa^{-1} \underset{m}{\hat{G}}^{ab} \tag{4}$$

$\hat{G}^{ab}$ is the truncated Einstein pseudotensor given by

$$\underset{m}{\hat{G}}^{ab} = G^{ab} - \underset{m}{L}_{ab} \tag{5}$$

where $\underset{m}{L}_{ab}$ is the linear part of the Einstein tensor G^{ab} for the metric $\underset{m}{\gamma}_{ab}$ and K^{ab}_{rs} the operator defined by

$$K^{ab}_{rs} = -\delta_{ar}\delta_{bs}J + J(\delta_{ar}D_{bs} + \delta_{bs}D_{ar} - \delta_{ab}D_{rs})J \tag{6}$$

where D_a and D_{ab} indicate partial derivatives of the first and second order, respectively, while J is the inverse D'Alambertian operator defined by

$$Jf(\mathbf{x}, t) = -(4\pi)^{-1} \int f(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|) |\mathbf{x} - \mathbf{x}'|^{-1} \, d_3x' \tag{7}$$

Latin indices take the values 1, 2, 3, 4 and Greek indices 1, 2, 3.

In order to introduce approximation we must have some estimate of the orders of magnitude of the physical quantities involved in the problem under consideration. Having obtained this estimate in a common unit (the second), we express all these magnitudes in terms of a single parameter $k < 1$ of order of the rate mass-size for the material system (Synge, 1970). Assuming provisionally that

$$T^{ab} = O(k) \tag{8}$$

we then have from (2)

$$\underset{m}{\gamma_{ab}} = \underset{m-1}{\gamma_{ab}} + O(k^m), \qquad (m = 1, 2, \ldots, N) \tag{9a}$$

and

$$\underset{m}{\hat{G}^{ab}} = \underset{m-1}{\hat{G}^{ab}} + O(k^{m+1}), \qquad (m = 1, 2, \ldots, N) \tag{9b}$$

To estimate the error in the Einstein field equations we have the error tensor in Nth approximation defined by

$$\underset{N}{E^{ab}} = \underset{N}{L_{ab}} + \underset{N}{\hat{G}^{ab}} + \kappa T^{ab} \tag{10}$$

Then, if we terminate the sequence (2) at the Nth term, we can make this Nth term satisfy Einstein field equations with an order k^{N+1} error, i.w.,

$$\underset{N}{E} = O(k^{N+1})$$

by requiring T^{ab} to satisy the equations of motion in Nth approximation

$$\underset{N-1}{H^{ab}}{}_{,b} = 0 \tag{11}$$

These equations are equivalent to

$$T^{ab}{}_{\underset{N}{|}b} = O(k^{N+1}) \tag{12}$$

where the N below the stroke indicates that the covariant derivative is calculated with respect to the metric tensor in $(N-1)$th approximation. They are called Synge equations of motion in Nth approximation.

With this scheme we can derive in a systematic way approximate solutions of Einstein field equations and equations of motion to any degree of accuracy we wish. Note that for equations of motion in the Nth approximation we only need the metric components to $O(k^{N-1})$. Next we shall relax this requirement when we consider the separate components of these equations and the different orders of magnitude for the components of the energy tensor T^{ab}. In the present work we consider equations of motion in fourth approximation.

3. THE EQUATIONS OF MOTION

In accord with what has been said, the equations of motion in fourth approximation are

$$T^{ab}{}_{,b} + \kappa^{-1} \underset{3}{\hat{G}}{}^{ab}{}_{,b} = 0 \tag{13}$$

Then, assuming that

$$T^{\alpha\beta} = O(k^2), \qquad T^{\alpha 4} = O(k^{1/2}), \qquad T^{44} = O(k) \tag{14}$$

we have

$$\underset{3}{\gamma_{ab}} = \underset{2}{\gamma_{ab}} + O(k^3) \tag{15}$$

and

$$\underset{3}{\hat{G}}{}^{ab} = \hat{G}_2{}^{ab} + O(k^4) \tag{16}$$

where $\underset{2}{\gamma_{ab}}$ and $\underset{3}{\gamma_{ab}}$ are the metric deviations of second and third order, respectively, and $\underset{2}{\hat{G}}{}^{ab}$ and $\underset{3}{\hat{G}}{}^{ab}$ are the truncated Einstein pseudotensors associated with these deviations, respectively.

In order to calculate $\underset{3}{\hat{G}}{}^{ab}$ with an error $O(k^4)$ it is enough to dispose of the metric in second approximately. By equations (38) of Synge (1970), this metric may be written in the form

$$\underset{2}{g_{ab}} = \delta_{ab} + \underset{2}{\gamma_{ab}} \tag{17}$$

where

$$\underset{2}{\gamma_{\alpha\beta}} = 2(V - K_{\sigma\sigma} + V^2)\delta_{\alpha\beta} + 4K_{\alpha\beta} + E_{\alpha\beta} + O(k^3) \tag{18}$$

$$\underset{2}{\gamma_{\alpha 4}} = 4i(W_\alpha + F_\alpha) + O(k^{7/2}) \tag{19}$$

$$\underset{2}{\gamma_{44}} = -2(V + K_{\sigma\sigma} - V^2) + O(k^3) \tag{20}$$

and

$$V = -RT^{44} = O(k) \tag{21}$$

$$W_\mu = -iRT^{\mu 4} = O(k^{3/2}) \tag{22}$$

$$K_{\mu\nu} = RT^{\mu\nu} = O(k^2) \tag{23}$$

$$E_{\mu\nu} = -\frac{1}{\pi} R(V_{,\mu}V_{,\nu} + 2VV_{,\mu\nu}) = O(k^2) \tag{24}$$

$$\begin{aligned} F_\mu = -\frac{i}{4\pi} R[&-3V_{,\mu}V_{,4} + 2i(V_{,\sigma}W_{\sigma,\mu} - V_{,\mu\sigma}W_\sigma) \\ &+ 2i(W_\mu \Box V - V \Box W_\mu) = O(k^{5/2}) \end{aligned} \tag{25}$$

with

$$R = -4\pi J \tag{26}$$

Then $\underset{3}{\hat{G}}{}^{ab}$ is given by (McCrea, 1981)

$$\begin{aligned} \underset{3}{\hat{G}}{}^{ab} = M_{ab} &- \underset{2}{\gamma_{rs}} L_{rabs} + \tfrac{1}{2}\underset{2}{\gamma_{ab}} L^*_{rr} + \delta_{ab}\underset{2}{\gamma_{rs}} L^*_{rs} \\ &-(\gamma_{2ar}L^*_{rb} + \underset{2}{\gamma_{br}}L^*_{ra}) + \underset{2}{\gamma_{mn}}[rb, m][ra, n] \\ &-\tfrac{1}{2}\delta_{ab}\underset{2}{\gamma_{mn}}[rs, m][rs, n] - \underset{2}{\gamma_{rs}}M_{rabs} + \underset{2}{\gamma_{rp}}\underset{2}{\gamma_{ps}}L_{rabs} \\ &+\delta_{ab}(\underset{2}{\gamma_{rs}}M^*_{rs} - \underset{2}{\gamma_{rp}}\underset{2}{\gamma_{ps}}L^*_{rs} - \tfrac{1}{2}\underset{2}{\gamma_{rs}}\underset{2}{\gamma_{pq}}L_{rpqs}) \\ &-\underset{2}{\gamma_{ab}}(\tfrac{1}{2}M^*_{pp} + \underset{2}{\gamma_{rs}}L^*_{rs}) - (\underset{2}{\gamma_{br}}M_{ra} + \underset{2}{\gamma_{ar}}M_{rb}) \\ &+\underset{2}{\gamma_{rs}}(\underset{2}{\gamma_{ap}}L_{rbps} + \underset{2}{\gamma_{bp}}L_{raps}) - \tfrac{1}{2}\underset{2}{\gamma_{as}}\underset{2}{\gamma_{bs}}L^*_{rr}) \\ &+\underset{2}{\gamma_{ar}}\underset{2}{\gamma_{bs}}L^*_{rs} + \underset{2}{\gamma_{rs}}(\underset{2}{\gamma_{ar}}L^*_{bs} + \underset{2}{\gamma_{br}}L^*_{as}) + O(k^4) \end{aligned} \tag{27}$$

where

$$L_{abcd} = \tfrac{1}{2}(\underset{2}{\gamma_{ad,bc}} + \underset{2}{\gamma_{bc,ad}} - \underset{2}{\gamma_{ac,bd}} - \underset{2}{\gamma_{bd,ac}}) \tag{28}$$

$$M_{abcd} = [ad, m][bc, m] - [ac, m][bd, m] \tag{29}$$

$$L_{bs} = L_{mbcm} - \tfrac{1}{2}\delta_{bc}L_{mrrm}, \qquad L^*_{bc} = L_{mbcm} \tag{30}$$

$$M_{bc} = M_{mbcm} - \tfrac{1}{2}\delta_{bc}M_{mrrm}, \qquad M^*_{bc} = M_{mbcm} \tag{31}$$

($[ad, m]$ are the Christoffel symbols of the first kind). From here we have the following expressions for the terms needed for our purpose:

$$\begin{aligned}\hat{G}^{44} =& 3(V_{,\sigma})^2+4V\Box V-4W_{\sigma,\nu}(W_{\sigma,\nu}-W_{\nu,\sigma})\\ &-4iV_{,\sigma}W_{\sigma,4}-2V_{,\sigma}K_{\nu\nu,\sigma}-4V_{,\sigma}K_{\sigma\nu,\nu}\\ &-3(V_{,4})^2-8K_{\sigma\sigma}\Box V+4V_{,\sigma\nu}K_{\sigma\nu}+10V(V_{,\sigma})^2+4V^2\Box V\\ &+(\tfrac{3}{2}E_{\nu\nu,\sigma}-E_{\nu\sigma,\nu})V_{,\sigma}\\ &+V_{,\sigma\nu}E_{\sigma\nu}+E_{\sigma\sigma}\Box V+O(k^4)\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{G}^{\alpha 4} =& -6V_{,\alpha}V_{,4}+4iV_{,\sigma}W_{\sigma,\alpha}\\ &+4i(W_\alpha\Box V-V\Box W_\alpha)-4iV_{,\alpha\sigma}W_\sigma+O(k^{7/2})\end{aligned} \tag{33}$$

$$\begin{aligned}\hat{G}^{\alpha\beta} =& -2V_{,\alpha}V_{,\beta}+3(V_{,\sigma})^2\delta_{\alpha\beta}-4VV_{,\alpha\beta}+4V\Box V\delta_{\alpha\beta}\\ &+3(V_{,4})^2\delta_{\alpha\beta}-2(V_{,\alpha}K_{\sigma\sigma,\beta}+V_{,\beta}K_{\sigma\sigma,\alpha})\\ &+V_{,\sigma}(-8K_{\alpha\beta,\sigma}+4K_{\sigma\beta,\alpha}+4K_{\sigma\alpha,\beta}+4K_{\nu\nu,\sigma}\delta_{\alpha\beta})\\ &+8W_{\sigma,\alpha}W_{\sigma,\beta}+8W_{\alpha,\sigma}W_{\beta,\sigma}+4iV_{,4}(W_{\alpha,\beta}+W_{\beta,\alpha})\\ &+\delta_{\alpha\beta}(-12W_{\sigma,\nu}W_{\sigma,\nu}+4W_{\sigma,\nu}W_{\nu,\sigma}-4V_{,\sigma}K_{\sigma\nu,\nu}\\ &+2V_{,\sigma}K_{\nu\nu,\sigma}-4iV_{,4}W_{\sigma,\sigma}+4iV_{,\sigma}W_{\sigma,4})\\ &+4VV_{,44}\delta_{\alpha\beta}+16W_\sigma W_{\sigma,\alpha\beta}\\ &+\delta_{\alpha\beta}(2VK_{\nu\nu,\sigma\sigma}+2V_{,\sigma\sigma}K_{\nu\nu}-4V_{,\sigma\nu}K_{\sigma\nu}-8iW_\sigma V_{,\sigma 4})\\ &+4V(K_{\sigma\alpha,\beta\sigma}+K_{\sigma\beta,\alpha\sigma}-K_{\alpha\beta,\sigma\sigma}-K_{\sigma\sigma,\alpha\beta})\\ &+4(V_{,\alpha\sigma}K_{\beta\sigma}+V_{,\beta\sigma}K_{\alpha\sigma}-V_{,\alpha\beta}K_{\sigma\sigma})\\ &+4i(V_{,\alpha 4}W_\beta+V_{,\beta 4}W_\alpha)-4iV(W_{\alpha,\beta 4}+W_{\beta,\alpha 4})\\ &-8W_\sigma(W_{\alpha,\beta\sigma}+W_{\beta,\alpha\sigma})-16\delta_{\alpha\beta}W_\sigma\Box W_\sigma\\ &+8(W_\alpha\Box W_\beta+W_\beta\Box W_\alpha)-8V\Box K_{\alpha\beta}-4K_{\alpha\beta}\Box V\\ &+\delta_{\alpha\beta}(2V\Box K_{\sigma\sigma}+2K_{\sigma\sigma}\Box V)-6V(V_{,\sigma})^2\delta_{\alpha\beta}+4VV_{,\alpha}V_{,\beta}\\ &-82V_{,\alpha\beta}-\tfrac{1}{2}(V_{,\alpha}E_{\sigma\sigma,\beta}+V_{,\beta}E_{\sigma\sigma,\alpha}-2V_{,\sigma}E_{\alpha\beta,\sigma}\\ &+V_{,\sigma}(E_{\alpha\sigma,\beta}+E_{\beta\sigma,\alpha})+V_{,\sigma}(\tfrac{3}{2}E_{\nu\nu,\sigma}-E_{\nu\sigma,\nu})\delta_{\alpha\beta}\\ &-V_{,\sigma\nu}E_{\sigma\nu}\delta_{\alpha\beta}-V_{,\alpha\beta}E_{\sigma\sigma}+V_{,\alpha\sigma}E_{\beta\sigma}\\ &+V_{,\sigma\beta}E_{\alpha\sigma}+V(E_{\alpha\sigma,\beta\sigma}+E_{\beta\sigma,\alpha\sigma}-E_{\sigma\sigma,\alpha\beta})\\ &-E_{\alpha\beta}\Box V+\delta_{\alpha\beta}E_{\sigma\sigma}\Box V+O(k^4)\end{aligned} \tag{34}$$

Writing now the metric tensor $\underset{3}{g}_{ab}$ in the form

$$\underset{3}{g}_{ab} = \delta_{ab} + p_{ab} + q_{ab} + r_{ab} + O(k^4) \tag{35}$$

where

$$p_{ab} = O(k), \qquad q_{ab} = O(k^2), \qquad r_{ab} = 0(k^3) \tag{36}$$

then the second term of equations (13) may be written in the form

$$\underset{4}{X'_a} + \underset{4}{X''_a} + \underset{4}{X'''_a} + O(k^5) \tag{37}$$

with an error $O(k^5)$. Here X'_a, X''_a, and X'''_a are given by

$$\underset{4}{X'_a} = -\tfrac{1}{2} p_{nn,c} T^{ac} + (\tfrac{1}{2} p_{bc,a} - p_{ab,c}) T^{bc} = O(k^2) \tag{38}$$

$$\begin{aligned} \underset{4}{X''_a} &= (\tfrac{1}{4} p_{mn} p_{mn} - \tfrac{1}{2} q_{nn})_{,c} T^{ac} \\ &\quad + (p_{an} p_{bn,c} - \tfrac{1}{2} p_{an} p_{bc,n} + \tfrac{1}{2} q_{bc,a} - q_{ab,c}) T^{bc} = O(k^3) \end{aligned} \tag{39}$$

$$\begin{aligned} \underset{4}{X'''_a} &= -\tfrac{1}{2} r_{nn,c} T^{ac} + \tfrac{1}{2}(p_{dm} q_{dm})_{,c} T^{ac} - \tfrac{1}{2} p_{dl} p_{lh} p_{hd,c} T^{ac} \\ &\quad - r_{ab,c} T^{bc} + \tfrac{1}{2} r_{bc,a} T^{bc} + p_{am} q_{bm,c} T^{bc} \\ &\quad - \tfrac{1}{2}(p_{ah} q_{bc,h} + q_{ah} p_{bc,h}) T^{bc} + q_{ah} p_{bh,c} T^{bc} \\ &\quad - p_{am} p_{mh} p_{hb,c} T^{bc} + \tfrac{1}{2} p_{am} p_{mn} p_{bc,n} T^{bc} = O(k^4) \end{aligned} \tag{40}$$

and are what we may call the components of first, second, and third order, respectively, for the 4-force $\underset{4}{X}_a$ in the field (for their derivation see Appendix A). Now, writing (37) in the form

$$\underset{4}{X}_\mu = A_{\mu\beta\gamma} T^{\beta\gamma} + B_{\mu\beta 4} T^{\beta 4} + C_\mu T^{44} + O(k^5) \tag{41}$$

$$\underset{4}{X}_4 = A_{4\beta\gamma} T^{\beta\gamma} + B_{4\beta 4} T^{\beta 4} + C_4 T^{44} + O(k^5) \tag{42}$$

and taking into account (14), we see that the expressions needed for $A_{\mu\beta\gamma}$, $B_{\mu\beta 4}$, C_μ, $A_{4\beta\gamma}$, $B_{4\beta 4}$, and C_4 in (41) and (42) are

$$\begin{aligned} A_{\mu\beta\gamma} &= -\tfrac{1}{2}[(p_{dd} + q_{dd}) - \tfrac{1}{2} p_{dh} p_{dh}]_{,\gamma} \delta_{\mu\beta} \\ &\quad + (\delta_{\mu\alpha} - p_{\mu\alpha})(\tfrac{1}{2} p_{\beta\gamma,\alpha} - p_{\beta\alpha,\gamma}) + \tfrac{1}{2} q_{\beta\gamma,\mu} - q_{\beta\mu,\gamma} + O(k^3) \end{aligned} \tag{43}$$

$$\begin{aligned} B_{\mu\beta 4} &= -\tfrac{1}{2}(p_{dd} + q_{dd} - \tfrac{1}{2} p_{dh} p_{dh})_{,4} \delta_{\mu\beta} \\ &\quad + (\delta_{\mu\alpha} - p_{\mu\alpha}) p_{\beta 4,\alpha} - (\delta_{\mu\alpha} - p_{\mu\alpha})(p_{\beta\alpha,4} + p_{\alpha 4,\beta}) \\ &\quad + p_{\mu 4} p_{44,\beta} + q_{4\beta,\mu} - q_{\beta\mu,4} - q_{4\mu,\beta} + O(k^{7/2}) \end{aligned} \tag{44}$$

$$C_\mu = (\delta_{\mu\alpha} - p_{\mu\alpha} + p_{\mu\varepsilon}p_{\varepsilon\alpha} - q_{\mu\alpha})(\tfrac{1}{2}p_{44,\alpha} - p_{4\alpha,4}) + \tfrac{1}{2}p_{\mu 4}p_{44,4} + \tfrac{1}{2}q_{44,\mu} - q_{\mu 4,4} - \tfrac{1}{2}p_{\mu\alpha}q_{44,\alpha} + \tfrac{1}{2}r_{44,\mu} + O(k^4) \tag{45}$$

$$A_{4\beta\gamma} = -p_{4\alpha}(\tfrac{1}{2}p_{\beta\gamma,\alpha} - p_{\beta\alpha,\gamma}) + (1 - p_{44})(\tfrac{1}{2}p_{\beta\gamma,4} - p_{\beta 4,\gamma}) + \tfrac{1}{2}q_{\beta\gamma,4} - q_{\beta 4,\gamma} + O(k^3) \tag{46}$$

$$\begin{aligned} B_{4\nu 4} = {} & -\tfrac{1}{2}p_{dd,\beta} - \tfrac{1}{2}q_{dd,\beta} - \tfrac{1}{2}r_{dd,\beta} + \tfrac{1}{4}(p_{\eta\varepsilon}p_{\eta\varepsilon})_{,\beta} \\ & + \tfrac{1}{2}(p_{\varepsilon 4}p_{\varepsilon 4})_{,\beta} + \tfrac{1}{4}(p_{44}p_{44})_{,\beta} - \tfrac{1}{6}(p_{\xi\eta}p_{\eta\varepsilon}p_{\varepsilon\xi})_{,\beta} - \tfrac{1}{6}(p_{44}^3)_{,\beta} \\ & + \tfrac{1}{2}(p_{\eta\varepsilon}q_{\eta\varepsilon})_{,\beta} + \tfrac{1}{2}(p_{44}q_{44})_{,\beta} \\ & + (-p_{4\alpha} + p_{4\varepsilon}p_{\varepsilon\alpha} + p_{44}p_{4\alpha} - q_{4a})(p_{\beta 4,\alpha} - p_{\beta\alpha,4} - p_{4\alpha,\beta}) \\ & - (1 - p_{44} + p_{4\varepsilon}p_{\varepsilon 4} + p_{44}^2 - q_{44})p_{44,\beta} - (1 - p_{44})q_{44,\beta} - r_{44,\beta} + O(k^{7/2}) \end{aligned} \tag{47}$$

$$\begin{aligned} C_4 = {} & -\tfrac{1}{2}[(p_{dd} + q_{dd} + r_{dd}) + p_{dh}(-\tfrac{1}{2}p_{dh} + \tfrac{1}{3}p_{hl}p_{ld} - q_{dh})]_{,4} \\ & + (p_{4\alpha} + p_{4\varepsilon}p_{\varepsilon\alpha} + p_{44}p_{4\alpha} - q_{4\alpha})(\tfrac{1}{2}p_{44,\alpha} - p_{\alpha 4,4}) \\ & - \tfrac{1}{2}(1 - p_{44} - q_{44})p_{44,4} - \tfrac{1}{2}p_{44}^2 p_{44,4} \\ & - \tfrac{1}{2}p_{4\alpha}q_{44,\alpha} - \tfrac{1}{2}(1 - p_{44})q_{44,4} - \tfrac{1}{2}r_{44,4} + O(k^4) \end{aligned} \tag{48}$$

On the other hand, taking into account (21)-(25), from (18)-(20) and (32)-(34) we have

$$p_{\mu\nu} = 2(V - K_{\sigma\sigma})\delta_{\mu\nu} + 4K_{\mu\nu} \tag{49}$$

$$q_{\mu\nu} = 2V^2\delta_{\mu\nu} + E_{\mu\nu} \tag{50}$$

$$p_{\mu 4} = 4iW_\mu \tag{51}$$

$$q_{\mu 4} = 4iF_\mu \tag{52}$$

$$\begin{aligned} r_{44} = {} & -8W_\sigma^2 - 4\pi^{-1}R(W_{\sigma,\nu}W_{\nu,\sigma}) - 4i\pi^{-1}R(V_{,\sigma}W_{\sigma,4}) - 8R(T^{44}K_{\sigma\sigma}) \\ & - 8R(VT^{\sigma\sigma}) + 4VK_{\sigma\sigma} - \pi^{-1}R(V_{,4})^2 + 2\pi^{-1}R(V_{,\sigma\nu}K_{\sigma\nu}) - 4V^3 \\ & + (2\pi)^{-1}R(V_{,\sigma\nu}E_{\sigma\nu}) - 2\pi^{-1}R(VV_{,44}) \\ & - 2\pi^{-1}R(VK_{\sigma\nu,\sigma\nu}) - (2\pi)^{-1}R(VE_{\sigma\nu,\sigma\nu}) \\ & + \pi^{-1}R(VV_{,\sigma}^2) + 8R(T^{44}V^2) \end{aligned} \tag{53}$$

Then

$$A_{\beta\mu\gamma} = (-4V + 4K_{\sigma\sigma} - V^2 - 2N)_{,\gamma}\delta_{\beta\mu} + (V_{,\mu} - K_{\sigma\sigma,\mu})\delta_{\beta\gamma} + \tfrac{1}{2}(4K_{\beta\gamma} + E_{\beta\gamma})_{,\mu} - (4K_{\beta\mu} + E_{\beta\mu})_{,\gamma} + O(k^3) \tag{54}$$

$$B_{\mu\beta 4} = -4V_{,4}\delta_{\mu\beta} + 4K_{\sigma\sigma,4}\delta_{\mu\beta} - (V^2)_{,4}\delta_{\mu\beta} - 2N_{,4}\delta_{\mu\beta} + 4i(W_{\beta,\mu} - W_{\mu,\beta})$$
$$-8iV(W_{\beta,\mu} - W_{\mu,\beta}) - 4K_{\mu\beta,4} - E_{\mu\beta,4}$$
$$+4i(F_{\beta,\mu} - F_{\mu,\beta}) - 8iW_\mu V_{,\beta} + O(k^{7/2}) \qquad (55)$$

$$C_\mu = -V_{,\mu} + 4VV_{,\mu} - 4iW_{\mu,4} - K_{\sigma\sigma,\mu} + 4VK_{\sigma\sigma,\mu}$$
$$+4K_{\mu\alpha}V_{,\alpha} + 8iVW_{\mu,4} - 12V^2V_{,\mu}$$
$$+E_{\mu\alpha}V_{,\alpha} - 4iW_\mu V_{,4} - 4iF_{\mu,4} - 4(W_\sigma)^2_{,\mu}$$
$$+\{R[-2\pi^{-1}W_{\sigma,\nu}W_{\nu,\sigma} - 2i\pi^{-1}V_{,\sigma}W_{\sigma,4} - 4T^{44}K_{\sigma\sigma}$$
$$-4VT^{\sigma\sigma} - (2\pi)^{-1}(V_{,4})^2 + \pi^{-1}V_{,\sigma\nu}K_{\sigma\nu}$$
$$+(4\pi)^{-1}V_{,\sigma\nu}E_{\sigma\nu} - \pi^{-1}VV_{,44} - \pi^{-1}VK_{\sigma\nu,\sigma\nu}$$
$$-(4\pi)^{-1}VE_{\sigma\nu,\sigma\nu} + (2\pi)^{-1}VV^2_{,\sigma} + 4T^{44}V^2]\}_{,\mu} + O(k^4) \qquad (56)$$

$$A_{4\beta\gamma} = V_{,4}\delta_{\beta\gamma} - 4iW_{\beta,\gamma} + 4VV_{,4}\delta_{\beta\gamma} + 8i(W_\beta V_{,\gamma} - VW_{\beta,\gamma})$$
$$-4iV_{,\alpha}W_\alpha\delta_{\beta\gamma} + 2K_{\beta\gamma,4} + \tfrac{1}{2}E_{\beta\gamma,4} - K_{\sigma\sigma,4}\delta_{\beta\gamma}$$
$$-4iF_{\beta,\gamma} + O(k^3) \qquad (57)$$

$$B_{4\beta 4} = [-V^2 + 4K_{\sigma\sigma} - 2N + VE_{\sigma\sigma} + 8W^2_\sigma - 4VK_{\sigma\sigma} + 8V^3]_{,\beta}$$
$$+16W_\alpha(W_{\beta,\alpha} - W_{\alpha,\beta})$$
$$+8iV_{,4}W_\beta - \pi^{-1}\{R[2(W_{\sigma,\nu})^2 + V_{,\sigma}(K_{\nu\nu,\sigma} + 2K_{\sigma\nu,\nu}$$
$$-6iW_{\sigma,4} - \tfrac{3}{4}E_{\nu\nu,\sigma} + \tfrac{1}{2}W_{\nu\sigma,\nu}) + \Box V(4K_{\sigma\sigma} - 2V^2 - \tfrac{1}{2}E_{\sigma\sigma})$$
$$-6W_{\sigma,\nu} + W_{\nu,\sigma} - \tfrac{1}{2}(V_{,4})^2 - 16\pi T^{44}(K_{\sigma\sigma} - V^2)$$
$$+V_{,\sigma\nu}(2K_{\sigma\nu} + \tfrac{1}{2}E_{\sigma\nu}) - V(4V_{,44} + 4K_{\sigma\nu,\nu\sigma} + E_{\sigma\nu,\nu\sigma} + 3(V_{,\sigma})^2$$
$$+16\pi T^{\sigma\sigma}(]\}_{,\beta} + O(k^{7/2}) \qquad (58)$$

$$C_4 = -V_{,4} + 3K_{\sigma\sigma,4} - 2VV_{,4} - 2N_{,4} + 4iV_{,\alpha}W_\alpha + 2V^2V_{,4} - 32W_\mu W_{\mu,4}$$
$$+4iV_{,\alpha}F_\alpha + 4iW_\alpha K_{\sigma\sigma,\alpha} + (VE_{\sigma\sigma,4}) + 2(VK_{\sigma\sigma})_{,4} - 8iVV_{,\alpha}W_\alpha$$
$$+i[-12(W^2_\sigma)_{,t} + 6(VK_{\sigma\sigma})_{,t} - 6(V^3)_{,t} + \pi^{-1}\{R[2(W_{\sigma,\nu})^2 - 4W_{\sigma,\nu}W_{\nu,\sigma}$$
$$-12\pi T^{44}(K_{\sigma\sigma} - V^2) + V_{,\sigma}(-4iW_{\sigma,4} + K_{\nu\nu,\sigma} + 2K_{\sigma\nu,\nu}$$

$$-\tfrac{3}{4}E_{\nu\nu,\sigma}+\tfrac{3}{4}E_{\nu\sigma,\nu})+V(-12\pi T^{\sigma\sigma}-3V_{,44}-3K_{\sigma\nu,\sigma\nu}-\tfrac{3}{4}E_{\sigma\nu,\sigma\nu}$$
$$-\tfrac{7}{2}(V_{,\sigma})^2)+V_{,\sigma\nu}K_{\sigma\nu}+\tfrac{1}{4}V_{,\sigma\nu}E_{\sigma\nu}$$
$$+\Box V[4K_{\sigma\sigma}-2V^2-\tfrac{1}{2}E_{\sigma\sigma})]\}_{,t}]+O(k^4) \tag{59}$$

Substituting now (54)-(59) in (41) and (42) and carrying the resultant expressions to (13), having separated this into 3+1 equations, we have concluded the first step in obtaining the equations of motion in fourth approximation. As can be seen in (54)-(59), every component of the 4-force is given in terms of retarded potentials. Now, assuming Synge's stationary initial conditions on the energy tensor T^{ab} (Synge, 1970), these retarded potentials can be expanded in terms of instantaneous potentials, since the retarded ones are given by the action of the integral operator R on density functions of the form $f(\mathbf{x}, t)$ as in (7) and (26). Then we may expand all these functions in the form

$$f(\mathbf{x}', t')=f(\mathbf{x}', t)-|\mathbf{x}-\mathbf{x}'|D_t f(\mathbf{x}', t)+\frac{|\mathbf{x}-\mathbf{x}'|^2}{2!}D_t^2 f(\mathbf{x}', t)$$
$$-\frac{|\mathbf{x}-\mathbf{x}'|^3}{3!}D_t^3 f(\mathbf{x}', t)+\cdots \tag{60}$$

so that

$$Rf=I_0 f-I_1 D_t f+\frac{1}{2!}I_2 D_t^2 f-\frac{1}{3!}I_3 D_t^3 f+\cdots \tag{61}$$

where

$$I_n f=\int f(\mathbf{x}', t)|\mathbf{x}-\mathbf{x}'|^{n+1}\,d_3x' \tag{62}$$

By Synge's conditions the integrand in $I_n D_t^n f$ has compact support for every density function in (54)-(59), and consequently (62) is finite for all n. Let us now expand the potentials (21)-(25) as well as the retarded integrals appearing in (56), (58), and (59).

Taking into account the orders of magnitude for each density function and that $D_t \equiv , t \equiv \partial/\partial t$ raises a quantity $O(k^n)$ to $O(k^{n+1/2})$, then we shall only keep in every expansion the significant terms in order to obtain the equations of motion in the desired approximation. The calculations for each integral are performed in Appendix B.

According to (B13), for the potential V we have

$$V=\tilde{V}-I_1(T^{44}\tilde{V}_{,t}+Z)-\tfrac{1}{2}D_t^2 I_2 T^{44}-\tfrac{1}{3}D_t I_1(T^{\sigma\sigma}-\tfrac{1}{2}T^{44}\tilde{V})$$
$$+\tfrac{1}{3}D_t\int(x_\mu-x'_\mu)Y_\mu\,d_3x'-\tfrac{1}{6}D_t I_3(T^{44}\tilde{V}_{,t})_{,t}$$
$$-\tfrac{1}{24}D_t^4 I_4 T^{44}-\tfrac{1}{120}D_t^3 I_5(T^{\mu\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\mu})_{,\mu}+O(k^4) \tag{63}$$

where $\tilde{V} = -I_0 T^{44}$ is the instantaneous "Newtonian potential" associated to V, and

$$Y_\mu = T^{\sigma\sigma}\tilde{V}_{,\mu} - 4T^{\mu\nu}\tilde{V}_{,\nu} - 4iT^{4\nu}(\tilde{W}_{\mu,\nu} - \tilde{W}_{n,\mu} - \delta_{\mu\nu}\tilde{V}_{,t})$$
$$-T_{44}D_\mu(-\tfrac{1}{2}D_t^2 I_2 T^{44} - 2\tilde{V}^2 + \tilde{K}_{\sigma\sigma}) - 4T^{44}D_t\tilde{W}_\mu = O(k^3) \quad (64)$$

where $D_\mu \equiv {}_{,\mu} \equiv \partial/\partial x_\mu$, and

$$Z = -T^{\sigma\sigma}\tilde{V}_{,t} - 4T^{\mu\nu}\tilde{W}_{\mu,\nu} - iT^{\mu\nu}D_\nu(4\tilde{K}_{\sigma\sigma} - \tilde{V}^2 - 2\tilde{N})$$
$$-T^{44}D_t[3\tilde{K}_{\sigma\sigma} - 2\tilde{N} - \tilde{V}^2 + \tfrac{1}{2}D_t^2 T^{44} + I_1(T^{44}\tilde{V}_{,t})]$$
$$+2T^{44}\tilde{V}_{,\sigma}\tilde{W}_\sigma - \tfrac{1}{3}T^{44}D_t^2 I_1(T^{\sigma\sigma} - \tfrac{1}{2}T^{44}\tilde{V}) = O(k^{7/2}) \quad (65)$$

with

$$\tilde{W}_\mu = -I_0(iT^{4\mu}), \qquad \tilde{K}_{\mu\nu} = I_0 T^{\mu\nu}, \qquad \tilde{N} = -I_0(T^{44}\tilde{V}) \quad (66)$$

In order to obtain the equations of motion, we will need the derivatives with respect to the spatial coordinates and also with respect to time. Then

$$V_{,\mu} = \tilde{V}_{,\mu} - \frac{1}{2}D_\mu D_t^2 I_2 T^{44} - \frac{1}{4!}D_\mu D_t^4 I_4 T^{44} - \frac{1}{3!}D_\mu D_t I_3(T^{44}\tilde{V}_{,t})_{,t}$$
$$+\frac{1}{3}D_t\int Y_\mu\, d_3x' - \frac{1}{5!}D_\mu D_t^3 I_5(T^{\sigma\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\sigma})_{,\sigma} + O(k^4) \quad (67)$$

$$V_{,t} = \tilde{V}_{,t} + D_t I_1(-T^{44}\tilde{V}_{,t} - Z) - \frac{1}{2}D_t^3 I_2 T^{44} - \frac{1}{3}D_t^2 I_1(T^{\sigma\sigma} - \frac{1}{2}T^{44}\tilde{V})$$
$$+\frac{1}{3}D_t^2\int (x_\sigma - x'_\sigma)Y_\sigma\, d_3x' - \frac{1}{6}D_t I_3(T^{44}\tilde{V}_{,t})_{,t} - \frac{1}{4!}D_t^5 I_4 T^{44}$$
$$-\frac{1}{5!}D_t^4 I_5(T^{\sigma\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\sigma})_{,\sigma} + O(k^{9/2}) \quad (68)$$

As is demonstrated in (B17), the expansions for the potential and its derivatives are

$$W_\mu = \tilde{W}_\mu - I_1(Y_\mu) - \tfrac{1}{2}D_t^2 I_2(iT^{\mu 4}) + \tfrac{1}{6}D_t^3 I_3(iT^{\mu 4}) + O(k^{7/2}) \quad (69)$$

$$W_{\mu,\nu} = \tilde{W}_{\mu,\nu} - \tfrac{1}{2}D_\nu D_t^2 I_2(iT^{\mu 4}) + \tfrac{1}{6}D_\nu D_t^3 I_3(iT^{\mu 4}) + O(k^{7/2}) \quad (70)$$

$$W_{\mu,t} = \tilde{W}_{\mu,t} - D_t I_1(Y_\mu) - \tfrac{1}{2}D_t^3 I_2(iT^{\mu 4}) + \tfrac{1}{6}D_t^4 I_3(iT^{\mu 4}) + O(k^4) \quad (71)$$

For the potential (23) we have

$$K_{\mu\nu}=\tilde{K}_{\mu\nu}-D_tI_1T^{\mu\nu}+\tfrac{1}{2}D_t^2I_2T^{\mu\nu}-\tfrac{1}{6}D_t^3I_3T^{\mu\nu}+O(k^4) \tag{72}$$

$$K_{\mu\nu,\gamma}=\tilde{K}_{\mu\nu,\gamma}+\tfrac{1}{2}D_\gamma D_t^2I_2T^{\mu\nu}-\tfrac{1}{6}D_\gamma D_t^3I_3T^{\mu\nu}+O(k^4) \tag{73}$$

$$K_{\mu\nu,t}=\tilde{K}_{\mu\nu,t}-D_t^2I_1T^{\mu\nu}+\tfrac{1}{2}D_t^3I_2T^{\mu\nu}-\tfrac{1}{6}D_t^4I_3T^{\mu\nu}+O(k^{9/2}) \tag{74}$$

and, in a similar way, for (24) we have

$$E_{\beta\gamma}=-\frac{1}{\pi}[\tilde{E}_{\beta\gamma}-D_tI_1(\tilde{V}_{,\beta}\tilde{V}_{,\gamma}+2\tilde{V}\tilde{V}_{,\beta\gamma})]+O(k^3) \tag{75}$$

$$E_{\beta\gamma,\mu}=-\frac{1}{\pi}D_\mu\tilde{E}_{\beta\gamma}+O(k^3) \tag{76}$$

$$E_{\beta\gamma,4}=\frac{i}{\pi}[\tilde{E}_{\beta\gamma,t}-D_t^2I_1(\tilde{V}_{,\beta}\tilde{V}_{,\gamma}+2\tilde{V}\tilde{V}_{,\beta\gamma})]+O(k^{7/2}) \tag{77}$$

where

$$\tilde{E}_{\beta\gamma}=I_0(\tilde{V}_{,\beta}\tilde{V}_{,\gamma}+2\tilde{V}\tilde{V}_{,\beta\gamma}) \tag{78}$$

On the other hand, expanding (25) and calculating its derivatives gives

$$F_\mu=\frac{1}{4\pi}\{\tilde{F}_\mu-D_tI_1[3\tilde{V}_{,\mu}\tilde{V}_{,t}+2(\tilde{V}_{,\sigma}\tilde{W}_{\sigma,\mu}-\tilde{V}_{,\mu\sigma}\tilde{W}_\sigma)$$
$$+2(\tilde{W}_\mu\Box\tilde{V}-\tilde{V}\Box\tilde{W}_\mu)]\}+O(k^{7/2}) \tag{79}$$

$$F_{\mu,\nu}=\frac{1}{4\pi}\tilde{F}_{\mu,\nu}+O(k^{7/2}) \tag{80}$$

$$F_{\mu,t}=-\frac{i}{4\pi}\{i\tilde{F}_{\mu,t}-D_t^2I_1[-3\tilde{V}_{,\mu}\tilde{V}_{,4}+2i(\tilde{V}_{,\sigma}\tilde{W}_{\sigma,\mu}-\tilde{V}_{,\mu\sigma}\tilde{W}_\sigma)$$
$$+2i(\tilde{W}_\mu\Box\tilde{V}-\tilde{V}\Box\tilde{W}_\mu)]\}+O(k^4) \tag{81}$$

where

$$\tilde{F}_\mu=-iI_0[-3\tilde{V}_{,\mu}\tilde{V}_{,4}+2i(\tilde{V}_{,\sigma}\tilde{W}_{\sigma,\mu}-\tilde{V}_{,\mu\sigma}\tilde{W}_\sigma)+2i(\tilde{W}_\mu\Box\tilde{V}-\tilde{V}\Box\tilde{W}_\mu)] \tag{82}$$

We need also the expansions for the potential

$$N=4\pi J(T^{44}V) \tag{83}$$

and for its derivatives. These are

$$N=\tilde{\tilde{N}}+D_tI_1(T^{44}\tilde{\tilde{V}})-\tfrac{1}{2}D_t^2I_2(T^{44}\tilde{V})+\tfrac{1}{6}D_t^3I_3(T^{44}\tilde{V})+O(k^4) \tag{84}$$

$$N_{,\mu}=\tilde{\tilde{N}}_{,\mu}-\tfrac{1}{2}D_\mu D_t^2I_2(T^{44}\tilde{V})+\tfrac{1}{6}D_\mu D_t^3I_3(T^{44}\tilde{V})+O(k^4) \tag{85}$$

$$N_{,t}=\tilde{\tilde{N}}_{,t}+D_t^2I_1(T^{44}\tilde{\tilde{V}})-\tfrac{1}{2}D_t^3I_2(T^{44}\tilde{V})+\tfrac{1}{6}D_t^4I_3(T^{44}\tilde{V})+O(k^{9/2}) \tag{86}$$

where

$$\tilde{\tilde{N}}=-I_0\{T^{44}[\tilde{V}-\tfrac{1}{2}D_t^2I_2T^{44}-I_1(T^{44}\tilde{V}_{,t})-\tfrac{1}{3}D_tI_1(T^{\sigma\sigma}-\tfrac{1}{2}T^{44}\tilde{V})]\} \tag{87}$$

and

$$\tilde{\tilde{V}}=\tilde{V}-\tfrac{1}{2}D_t^2I_2T^{44} \tag{88}$$

We will consider the retarded integral that appears at the end of (56) as a potential, which we denote by $\mathscr{A}$, i.e.,

$$\begin{aligned}\mathscr{A}=R[&-2\pi^{-1}W_{\sigma,\nu}W_{\nu,\sigma}-2i\pi^{-1}V_{,\sigma}W_{\sigma,4}-4T^{44}K_{\sigma\sigma}\\&-4VT^{\sigma\sigma}-(2\pi)^{-1}(V_{,4})^2+\pi^{-1}V_{,\sigma\nu}K_{\sigma\nu}\\&+(4\pi)^{-1}V_{,\sigma\nu}E_{\sigma\nu}-\pi^{-1}VV_{,44}-\pi^{-1}VK_{\sigma\nu,\sigma\nu}\\&-(4\pi)^{-1}VE_{\sigma\nu,\sigma\nu}+(2\pi)^{-1}V(V_{,\sigma})^2+4T^{44}V^2]\end{aligned} \tag{89}$$

Its expansion in instantaneous potentials is

$$\begin{aligned}\mathscr{A}=I_0\Big\{&-2\pi^{-1}\tilde{W}_{\sigma,\nu}\tilde{W}_{\nu,\sigma}-2i\pi^{-1}\tilde{V}_{,\sigma}\tilde{W}_{\sigma,4}-4T^{44}(\tilde{K}_{\sigma\sigma}-D_tI_1T^{\sigma\sigma})\\&-4T^{\sigma\sigma}\tilde{V}-(2\pi)^{-1}(\tilde{V}_{,4})^2+\pi^{-1}\tilde{V}_{,\sigma\nu}(\tilde{K}_{\sigma\nu}-D_tI_1T^{\sigma\nu})\\&+(4\pi)^{-1}\tilde{V}_{,\sigma\nu}\left[\tilde{E}_{\sigma\nu}\left(-\frac{1}{\pi}\right)+\left(\frac{1}{\pi}\right)D_tI_1(\tilde{V}_{,\sigma}\tilde{V}_{,\nu}+2V\tilde{V}_{,\sigma\nu})\right]-\pi^{-1}\tilde{V}\tilde{V}_{,44}\\&-\pi^{-1}\tilde{V}\tilde{K}_{\sigma\nu,\nu\sigma}+(4\pi)^{-1}\pi^{-1}\tilde{V}\tilde{E}_{\sigma\nu,\nu\sigma}+(2\pi)^{-1}\tilde{V}(\tilde{V}_{,\sigma})^2+4T^{44}\tilde{V}^2\Big\}\\&-D_tI_1[-2\pi^{-1}\tilde{W}_{\sigma,\nu}\tilde{W}_{\nu,\sigma}-2i\pi^{-1}\tilde{V}_{,\sigma}\tilde{W}_{\sigma,4}-4T^{44}\tilde{K}_{\sigma\sigma}\\&-4\tilde{V}T^{\sigma\sigma}-(2\pi)^{-1}(\tilde{V}_{,4})^2\\&+\pi^{-1}\tilde{V}_{,\sigma\nu}\tilde{K}_{\sigma\nu}+(4\pi)^{-1}\tilde{V}_{,\sigma\nu}\tilde{E}_{\sigma\nu}-\pi^{-1}\tilde{V}\tilde{V}_{,44}-\pi^{-1}\tilde{V}\tilde{K}_{\sigma\nu,\nu\sigma}\\&-(4\pi)^{-1}\tilde{V}\tilde{E}_{\sigma\nu,\sigma\nu}+(2\pi)^{-1}\tilde{V}(\tilde{V}_{,\sigma})^2+4T^{44}\tilde{V}^2]+O(k^4)\end{aligned} \tag{90}$$

and we also have

$$\mathscr{A}_{,\mu}=\tilde{\mathscr{A}}_{,\mu}+O(k^4) \tag{91}$$

where $\tilde{\mathscr{A}}$ is the first term of the two into which $\mathscr{A}$ is divided. On the other hand, from (58) we can define the retarded potential

$$\begin{aligned}\mathscr{B} = R(b) \equiv R\{&2(W_{\sigma,\nu})^2 + V_{,\sigma}[K_{\nu\nu,\sigma} + 2K_{\sigma\nu,\nu} - 6iW_{\sigma,4} - \tfrac{1}{2}(\tfrac{3}{2}E_{\nu\nu,\sigma} - E_{\nu\sigma,\nu})] \\ &+\square V(4K_{\sigma\sigma} - 2V^2 - \tfrac{1}{2}E_{\sigma\sigma}) - 6W_{\sigma,\nu}W_{\nu,\sigma} - \tfrac{1}{2}(V_{,4})^2 \\ &-16\pi T^{44}(K_{\sigma\sigma} - V^2) + V_{,\sigma\nu}(2K_{\sigma\nu} + \tfrac{1}{2}E_{\sigma\nu}) \\ &- V[4V_{,44} + 4K_{\sigma\nu,\nu\sigma} + E_{\sigma\nu,\nu\sigma} + 3(V_{,\sigma})^2 + 16\pi T^{\sigma\sigma}]\}\end{aligned} \tag{92}$$

Then, expanding the density b, retaining terms up to $O(k^4)$, and finally expanding the retarded integral operator R, we obtain

$$\begin{aligned}\mathscr{B} = \tilde{\mathscr{B}} = I_0\{&2(\tilde{W}_{\sigma,\nu})^2 + \tilde{V}_{,\sigma}[\tilde{K}_{\nu\nu,\sigma} + 2\tilde{K}_{\sigma\nu,\nu} - 6i\tilde{W}_{\sigma,4} \\ &+\tfrac{1}{2}\pi^{-1}(\tfrac{3}{2}\tilde{E}_{\nu\nu,\sigma} - \tilde{E}_{\nu\sigma,\nu})] + \square\tilde{V}[4\tilde{K}_{\sigma\sigma} - 2(\tilde{V})^2 + (2\pi)^{-1}\tilde{E}_{\sigma\sigma}] \\ &-6\tilde{W}_{\sigma,\nu}\tilde{W}_{\nu,\sigma} - \tfrac{1}{2}(\tilde{V}_{,4})^2 - 16\pi T^{44}(\tilde{K}_{\sigma\sigma} - \tilde{V}^2) \\ &+\tilde{V}_{,\sigma\nu}[2\tilde{K}_{\sigma\nu} - (2\pi)^{-1}\tilde{E}_{\sigma\nu}] \\ &-\tilde{V}[4\tilde{V}_{,44} + 4\tilde{K}_{\sigma\nu,\nu\sigma} - \pi^{-1}\tilde{E}_{\sigma\nu,\nu\sigma} + 3(\tilde{V}_{,\sigma})^2 + 16\pi T^{\sigma\sigma}]\} + O(k^{7/2})\end{aligned} \tag{93}$$

Following an analogous process for the retarded integral that appears in (59), we have

$$\begin{aligned}\mathscr{C} = I_0\Bigg\{&2(\tilde{W}_{\sigma,\nu})^2 - 4\tilde{W}_{\sigma,\nu}\tilde{W}_{\nu,\sigma} - 12\pi T^{44}(\tilde{K}_{\sigma\sigma} - D_t I_1 T^{\sigma\sigma} - \tilde{V}^2) \\ &+\tilde{V}_{,\sigma}\left[-4i\tilde{W}_{\sigma,4} + \tilde{K}_{\nu\nu,\sigma} + 2\tilde{K}_{\sigma\nu,\nu} + \frac{3}{4\pi}(\tilde{E}_{\nu\nu,\sigma} - \tilde{E}_{\nu\sigma,\nu})\right] \\ &+\tilde{V}\left[-12\pi T^{\sigma\sigma} - 3\tilde{V}_{,44} - 3\tilde{K}_{\sigma\nu,\sigma\nu} + \frac{3}{4\pi}\tilde{E}_{\sigma\nu,\nu\sigma} - \frac{7}{2}(\tilde{V}_{,\sigma})^2\right] \\ &+\tilde{V}_{,\sigma\nu}[\tilde{K}_{\sigma\nu} - D_t I_1 T^{\sigma\nu} - (4\pi)^{-1}\tilde{V}_{,\sigma\nu}\tilde{E}_{\sigma\nu} \\ &+(4\pi)^{-1}D_t I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\nu} + 2\tilde{V}\tilde{V}_{,\sigma\nu})] \\ &+\square\tilde{V}[4\tilde{K}_{\sigma\sigma} - 4D_t I_1 T^{\sigma\sigma} - 2\tilde{V}^2 + (2\pi)^{-1}\tilde{E}_{\sigma\sigma} \\ &-(2\pi)^{-1}D_t I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma} + 2\tilde{V}\tilde{V}_{,\sigma\sigma})]\Bigg\} \\ -D_t I_1\Bigg\{&2(\tilde{W}_{\sigma,\nu})^2 - 4\tilde{W}_{\sigma,\nu}\tilde{W}_{\nu,\sigma} - 12\pi T^{44}(\tilde{K}_{\sigma\sigma} - \tilde{V}^2) \\ &+\tilde{V}_{,\sigma}\left[-4\tilde{W}_{\sigma,t} + \tilde{K}_{\nu\nu,\sigma} + 2\tilde{K}_{\sigma\nu,\nu} + \frac{3}{4\pi}(\tilde{E}_{\nu\nu,\sigma} - \tilde{E}_{\nu\sigma,\nu})\right]\end{aligned}$$

$$+\tilde{V}\left[-12\pi T^{\sigma\sigma}+3\tilde{V}_{,tt}-3\tilde{K}_{\sigma\nu,\nu\sigma}+\frac{3}{4\pi}\tilde{E}_{\sigma\nu,\nu\sigma}-\frac{7}{2}(\tilde{V}_{,\sigma})^2\right]$$

$$+\tilde{V}_{,\sigma\nu}[\tilde{K}_{\sigma\nu}-(4\pi)^{-1}\tilde{E}_{\sigma\nu}]+\Box\tilde{V}\left(4\tilde{K}_{\sigma\sigma}-2\tilde{V}^2+\frac{1}{2\pi}\tilde{E}_{\sigma\sigma}\right)\Bigg\}+O(k^4) \tag{94}$$

and we also have

$$D_t\mathscr{C}=D_t\tilde{\mathscr{C}}-D_t^2I_1(\tilde{c})+O(k^{9/2}) \tag{95}$$

where $\mathscr{C}$ is the first term of the two into which $\mathscr{C}$ is divided, and $\tilde{c}$ is the integrand between keys in the other. Now using the expressions obtained in (63)-(95), we have for the coefficients of the components for the 4-force the following expressions:

(a) From (54)

$$A_{\beta\mu\gamma}=[-4\tilde{\tilde{V}}+4\tilde{K}_{\sigma\sigma}-\tilde{V}^2-2\tilde{N}]_{,\gamma}\delta_{\beta\mu}+(\tilde{V}_{,\mu}-\tfrac{1}{2}D_\mu D_t^2I_2T^{44}-\tilde{K}_{\sigma\sigma,\mu})\delta_{\beta\gamma}$$

$$+\frac{1}{2}\left(4\tilde{K}_{\beta\gamma}-\frac{1}{\pi}\tilde{E}_{\beta\gamma}\right)_{,\mu}-\left(4\tilde{K}_{\beta\mu}-\frac{1}{\pi}\tilde{E}_{\beta\mu}\right)_{,\gamma}+O(k^3) \tag{96}$$

(b) From (55)

$$B_{\mu\beta4}=i\{4[\tilde{V}_{,t}-D_tI_1(T^{44}\tilde{V}_{,t})-\tfrac{1}{2}D_t^3I_2T^{44}-\tfrac{1}{3}D_t^2I_1(T^{\sigma\sigma}+\tfrac{1}{2}T^{44}\tilde{V})]$$

$$-4(\tilde{K}_{\sigma\sigma,t}-D_t^2I_1T^{\sigma\sigma})+(\tilde{V}^2)_{,t}+2[\tilde{N}_{,t}+D_t^2I_1(T^{44}\tilde{V})]\}\delta_{\mu\beta}$$

$$+4i(1-2\tilde{V})(\tilde{W}_{\beta,\mu}-\tilde{W}_{\mu,\beta})+2(D_\mu D_t^2I_2T^{4\beta}-D_\beta D_t^2I_2T^{4\mu})$$

$$-\tfrac{2}{3}(D_\mu D_t^3I_3T^{4\beta}-D_\beta D_t^3I_3T^{4\mu}+4i(\tilde{K}_{\mu\beta,t}-D_t^2I_1T^{\mu\beta})$$

$$-\frac{i}{\pi}[\tilde{E}_{\mu\beta,t}-D_t^2I_1(\tilde{V}_{,\mu}\tilde{V}_{,\beta}+2\tilde{V}\tilde{V}_{,\mu\beta})]+\frac{i}{\pi}(\tilde{F}_{\beta,\mu}-\tilde{F}_{\mu,\beta})$$

$$-8i\tilde{W}_\mu\tilde{V}_{,\beta}+O(k^{7/2}) \tag{97}$$

(c) From (56)

$$C_\mu=-\tilde{V}_{,\mu}+\frac{1}{2}D_\mu D_t^2I_2T^{44}-\frac{1}{3}D_tI_1Y_\mu+\frac{1}{3!}D_\mu D_tI_3(T^{44}\tilde{V}_{,t})_{,t}$$

$$+\frac{1}{4!}D_\mu D_t^4I_4T^{44}+\frac{1}{5!}D_\mu D_t^3I_5(T^{\sigma\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\sigma})_{,\sigma}$$

$$+4\tilde{V}\tilde{V}_{,\mu}-2\tilde{V}_{,\mu}D_t^2I_2T^{44}-\frac{4}{3}\tilde{V}_{,\mu}D_tI_1(T^{\sigma\sigma}-\tfrac{1}{2}T^{44}\tilde{V})$$

$$-4\tilde{V}_{,\mu}I_1(T^{44}\tilde{V}_{,t})-2\tilde{V}D_\mu D_t^2I_2T^{44}-4\tilde{W}_{\mu,t}$$

$$+4D_tI_1Y_\mu+2D_t^3I_2(iT^{4\mu})-\frac{2}{3}D_t^4I_3(iT^{4\mu})$$

$$-\tilde{K}_{\sigma\sigma,\mu}-\frac{1}{2}D_\mu D_t^2I_2T^{\sigma\sigma}+\frac{1}{6}D_\mu D_t^3I_3T^{\sigma\sigma}$$

$$+4\tilde{V}\tilde{K}_{\sigma\sigma,\mu}+4\tilde{K}_{\mu\sigma}\tilde{V}_{,\sigma}-4\tilde{V}_{,\sigma}D_tI_1T^{\mu\sigma}+8\tilde{V}\tilde{W}_{\mu,t}-12\tilde{V}^2\tilde{V}_{,\mu}-\pi^{-1}\tilde{E}_{\mu\sigma}\tilde{V}_{,\sigma}$$

$$+\pi^{-1}\tilde{V}_{,\sigma}D_tI_1(\tilde{V}_{,\mu}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\mu\sigma})-4\tilde{W}_\mu\tilde{V}_{,t}-\frac{1}{\pi}\tilde{F}_{\mu,t}+\frac{1}{\pi}D_t^2I_1[3\tilde{V}_{,\mu}\tilde{V}_{,t}$$

$$+2(\tilde{V}_{,\sigma}\tilde{W}_{\sigma,\mu}-\tilde{V}_{,\mu\sigma}\tilde{W}_\sigma)+2(\tilde{W}_\mu\Box\tilde{V}-\tilde{V}\Box\tilde{W}_\mu)]$$

$$-4(\tilde{W}_\sigma^2)_{,\mu}+\Box\mathscr{A}_{,\mu}+O(k^4) \tag{98}$$

(d) From (57)

$$A_{4\beta\gamma}=-i\left[\tilde{V}_{,t}-\frac{1}{2}D_t^3I_2T^{44}-D_tI_1(T^{44}\tilde{V}_{,t})\right.$$

$$\left.-\frac{1}{3}D_t^2I_1\left(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}\right)\right]\delta_{\beta\gamma}$$

$$-4i\left[\tilde{W}_{\beta,\gamma}-\frac{1}{2}D_\gamma D_t^2I_2(iT^{4\beta})+\frac{1}{6}D_\gamma D_t^3I_3(iT^{4\beta})\right]$$

$$-4i\tilde{V}\tilde{V}_{,t}\delta_{\beta\gamma}+8i(\tilde{W}_\beta\tilde{V}_{,\gamma}-\tilde{V}\tilde{W}_{\beta,\gamma})-4i\tilde{V}_{,\sigma}\tilde{W}_\sigma\delta_{\beta\gamma}$$

$$-2i(\tilde{K}_{\beta\gamma,t}-D_t^2I_1T^{\beta\gamma})$$

$$+\frac{i}{2\pi}[\tilde{E}_{\beta\gamma,t}-D_t^2I_1(\tilde{V}_{,\beta}\tilde{V}_{,\gamma}+2\tilde{V}\tilde{V}_{,\beta\gamma})]-i(\tilde{K}_{\sigma\sigma,t}-D_t^2I_1T^{\sigma\sigma})\delta_{\beta\gamma}$$

$$-\frac{i}{\pi}\tilde{F}_{\beta,\gamma}+O(k^{7/2}) \tag{99}$$

(e) From (58)

$$B_{4\beta 4}=-(\tilde{V})^2{}_{,\beta}+(\tilde{V}D_t^2I_2T^{44})_{,\beta}+2\tilde{V}_{,\beta}I_1\left[T^{44}\tilde{V}_{,t}+\frac{1}{3}D_t\left(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}\right)\right]$$

$$+4\tilde{K}_{\sigma\sigma,\beta}+2D_\beta D_t^2I_2T^{\sigma\sigma}-\frac{2}{3}D_\beta D_t^3I_3T^{\sigma\sigma}-2\tilde{\tilde{N}}_{,\beta}$$

$$+D_\beta D_t^2I_2(T^{44}\tilde{V})-\frac{1}{3}D_\beta D_t^3I_3(T^{44}\tilde{V})$$

$$-\frac{1}{\pi}(\tilde{V}\tilde{E}_{\sigma\sigma})_{,\beta}+\frac{1}{\pi}\tilde{V}_{,\beta}D_tI_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})$$

$$+8(\tilde{W}_\sigma^2)_{,\beta}-4(\tilde{V}\tilde{K}_{\sigma\sigma})_{,\beta}+4\tilde{V}_{,\beta}D_tI_1T^{\sigma\sigma}+8(\tilde{V}^3)_{,\beta}$$
$$+16\tilde{W}_\sigma(\tilde{W}_{\beta,\sigma}-\tilde{W}_{\sigma,\beta})+8\tilde{V}_{,t}\tilde{W}_\beta-\pi^{-1}\tilde{\mathscr{B}}_{,\beta}+O(k^4) \tag{100}$$

(f) From (59)

$$C_4=i\bigg\{\tilde{V}_{,t}-\frac{1}{2}D_t^3I_2T^{44}-D_tI_1(T^{44}\tilde{V}_{,t})-\frac{1}{3}D_t^2I_1\bigg(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}\bigg)$$
$$-\frac{1}{6}D_tI_3(T^{44}\tilde{V}_{,t})_{,t}-\frac{1}{4!}D_t^5I_4T^{44}-D_tI_1Z+\frac{1}{3}D_t^2I_1[(x_\sigma-x'_\sigma)Y_\sigma]$$
$$-\frac{1}{5!}D_t^4I_5(T^{\sigma\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\sigma})_{,\sigma}\bigg\}-3i[\tilde{K}_{\sigma\sigma,t}-D_t^2I_1T^{\sigma\sigma}]-\frac{3}{2}iD_t^3I_2T^{\sigma\sigma}$$
$$+\frac{i}{2}D_t^4I_3T^{\sigma\sigma}+2i\tilde{V}\tilde{V}_{,t}-i\tilde{V}D_t^3I_2T^{44}$$
$$-2i\tilde{V}D_tI_1(T^{44}\tilde{V}_{,t})-\frac{2i}{3}\tilde{V}D_t^2I_1\bigg(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}\bigg)$$
$$-iD_t^2(I_2T^{44})\tilde{V}_{,t}-\frac{2i}{3}D_tI_1\bigg(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}\bigg)\tilde{V}_{,t}$$
$$+2i\bigg[\tilde{\tilde{N}}_{,t}+D_t^2I_1(T^{44}\tilde{\tilde{V}})-\frac{1}{2}D_t^3I_2(T^{44}\tilde{V})+\frac{1}{6}D_t^4I_3(T^{44}\tilde{V})\bigg]$$
$$+4i\tilde{V}_{,\sigma}\tilde{W}_\sigma-2i\tilde{V}_{,\sigma}D_t^2I_2(iT^{4\sigma})-4i\tilde{V}_{,\sigma}I_1(Y_\sigma)$$
$$+\frac{2i}{3}\tilde{V}_{,\sigma}D_t^3I_3(iT^{4\sigma})-2i\tilde{W}_\sigma D_\sigma D_t^2I_2T^{44}-20i\tilde{V}^2\tilde{V}_{,t}+8i\tilde{W}_\sigma\tilde{W}_{\sigma,t}$$
$$+\frac{i}{\pi}\tilde{V}_{,\sigma}\{\tilde{F}_\sigma-D_tI_1[3\tilde{V}_{,\sigma}\tilde{V}_{,t}+2(\tilde{V}_{,\nu}\tilde{W}_{\nu,\sigma}-\tilde{V}_{,\nu\sigma}\tilde{W}_\nu)$$
$$+2(\tilde{W}_\sigma\,\Box\,\tilde{V}-\tilde{V}\,\Box\,\tilde{W}_\sigma)]\}+4i\tilde{W}_\sigma\tilde{K}_{\nu\nu,\sigma}$$
$$+\frac{i}{\pi}\tilde{V}_{,t}\tilde{E}_{\sigma\sigma}-\frac{i}{\pi}\tilde{V}_{,t}D_tI_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})+\frac{i}{\pi}\tilde{V}\tilde{E}_{\sigma\sigma,t}$$
$$-\frac{i}{\pi}\tilde{V}D_t^2I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})+4i\tilde{V}_{,t}\tilde{K}_{\sigma\sigma}$$
$$-4i\tilde{V}_{,t}D_tI_1T^{\sigma\sigma}+4i\tilde{V}\tilde{K}_{\sigma\sigma,t}-4i\tilde{V}D_t^2I_1T^{\sigma\sigma}-8i\tilde{V}\tilde{V}_{,\sigma}\tilde{W}_\sigma$$
$$+i\pi^{-1}[D_t\tilde{\mathscr{C}}-D_t^2I_1(\tilde{C})]+O(k^{9/2}) \tag{101}$$

At this stage we have at our disposal in (96)-(101) the expressions for the components of the 4-force in terms of instantaneous potentials. Taking into account (37), (41), and (42), it is possible now to determine the diverse relativistic contributions of each order up to $O(k^5)$ both for the classical force in the equations of motion as well as the equation of continuity. But, in order to be able to do this, we must do the final step, i.e., we now have to choose an Eulerian formalism for the description of the motion of the material system or, what is the same, we must choose an Eulerian decomposition of the material energy tensor T^{ab}. In this way, it also will be possible to verify papers and, in particular, the equations obtained in Synge's paper.

The Eulerian formalism that we adopt is

$$T^{\mu\nu} = \rho u_\mu u_\nu - S_{\mu\nu} \tag{102}$$

$$T^{\mu 4} = i\rho u_\mu \tag{103}$$

$$T^{44} = -\rho \tag{104}$$

These equations express the ten components T^{ab} of the material energy tensor in terms of the Eulerian variables ρ (density), $S_{\mu\nu}$ (stress), and m_μ (velocity). Equivalently, we may write (102)-(104) in the form

$$\rho = -T^{44} \tag{105}$$

$$u_\mu = iT^{\mu 4}/T^{44} \tag{106}$$

$$S_{\mu\nu} = T^{\mu 4}T^{\nu 4}/T^{44} - T^{\mu\nu} \tag{107}$$

and regard this as a definition of the Eulerian variables in terms of T^{ab}. With these variables we have for the first term of equations (13)

$$T^{\mu b}{}_{,b} = \rho \dot{u}_\mu + u_\mu(\dot{\rho} + \rho\theta) - S_{\mu\nu,\nu} \tag{108}$$

$$-iT^{4b}{}_{,b} = \dot{\rho} + \rho\theta \tag{109}$$

where $\theta = u_{b,b} = u_{\mu,\mu}$ is the expansion and the dot means total derivative with respect to time. Then, taking into account (37), (41), (42), and (96)-(104), the equations of motion (13) take the final form

$$\rho \dot{u}_\mu + u_\mu(\dot{\rho} + \rho\theta) - S_{\mu\nu,\nu} = \rho \tilde{V}_{,\mu} + Y_\mu + Y'_\mu + Y''_\mu + O(k^5) \tag{110}$$

$$\dot{\rho} + \rho\theta = -\rho \tilde{V}_{,t} + Z_1 + Z_2 + Z_3 + Z_4 + O(k^{11/2}) \tag{111}$$

where

$$Y_\mu = (\rho u^2 - S_{\sigma\sigma})\tilde{V}_{,\mu} - 4(\rho u_\mu u_\gamma - S_{\mu\gamma})\tilde{V}_{,\gamma} + 4\rho u_\beta(\tilde{W}_{\mu,\beta} - \tilde{W}_{\beta,\mu} - \delta_{\mu\beta}\tilde{V}_{,t})$$
$$+ \rho D_\mu(\tfrac{1}{2}D_t^2 I_2\rho - 2\tilde{V}^2 + \tilde{K}_{\sigma\sigma}) + 4\rho \tilde{W}_{\mu,t} \tag{112}$$

$$
\begin{aligned}
Y'_\mu = {} & (\rho u^2 - S_{\sigma\sigma})\left(\frac{1}{2}D_t^2 I_2\rho - \tilde{K}_{\nu\nu}\right)_{,\mu} + (\rho u_\beta u_\gamma - S_{\beta\gamma})\Bigg[\left(2\tilde{K}_{\beta\gamma} - \frac{1}{2\pi}\tilde{E}_{\beta\gamma}\right)_{,\mu} \\
& -2\left(2\tilde{K}_{\beta\mu} - \frac{1}{2\pi}\tilde{E}_{\beta\mu}\right)_{,\gamma} + \delta_{\mu\beta}(-2D_t^2 I_2\rho + 4\tilde{K}_{\sigma\sigma} - \tilde{V}^2 - 2\tilde{N})_{,\gamma}\Bigg] \\
& +\rho u_\beta \Bigg\{ 8[(\tilde{W}_{\beta,\mu} - \tilde{W}_{\mu,\beta})\tilde{V} + \tilde{W}_\mu \tilde{V}_{,\beta}] - 2D_t^2[D_\mu I_2(\rho u_\beta) - D_\beta I_2(\rho u_\mu)] \\
& -\frac{1}{\pi}(\tilde{F}_{\beta,\mu} - \tilde{F}_{\mu,\beta}) - 2\left(2\tilde{K}_{\mu\beta} - \frac{1}{2\pi}\tilde{E}_{\mu\beta}\right)_{,t} \\
& +\delta_{\mu\beta}(-2D_t^2 I_2\rho + 4\tilde{K}_{\sigma\sigma} - \tilde{V}^2 - 2\tilde{N})_{,t}\Bigg\} \\
& +\rho D_\mu\left[\frac{1}{24}D_t^4 I_4\rho - 2\tilde{V}D_t^2 I_2\rho + \frac{1}{2}D_t^2 I_2(\rho u^2 - S_{\sigma\sigma}) + 4(\tilde{V}^3 + \tilde{W}_\sigma^2) - \tilde{\mathscr{A}}\right] \\
& -\rho\tilde{V}_{,\sigma}\left(4\tilde{K}_{\mu\sigma} - \frac{1}{\pi}\tilde{E}_{\mu\sigma}\right) + \rho\left[2D_t^2 I_2(\rho u_\mu) + \frac{1}{\pi}\tilde{F}_\mu\right]_{,t} \\
& -4\rho\tilde{V}(\tilde{K}_{\sigma\sigma,\mu} + 2\tilde{W}_{\mu,t}) + 4\rho\tilde{W}_\mu\tilde{V}_{,t}
\end{aligned} \tag{113}
$$

$$
\begin{aligned}
Y''_\mu = {} & -4\rho\tilde{V}_{,\mu}I_1(\rho\tilde{V}_{,t}) + \frac{2}{3}\rho u_\beta D_t\Bigg\{ D_\mu D_t^2 I_3(\rho u_\beta) - D_\beta D_t^2 I_3(\rho u_\mu) \\
& +6D_t I_1(\rho u_\mu u_\beta - S_{\mu\beta}) - \frac{1}{\pi}D_t I_1(\tilde{V}_{,\mu}\tilde{V}_{,\beta} + 2\tilde{V}\tilde{V}_{,\mu\beta}) \\
& -\delta_{\mu\beta}\left[6I_1(\rho\tilde{V}_{,t}) + 4D_t I_1\left(\rho u^2 - S_{\sigma\sigma} - \frac{1}{2}\rho\tilde{V}\right)\right]\Bigg\} \\
& +\frac{1}{3}\rho D_t\Bigg\{\frac{1}{2}D_\mu D_t I_3(\rho\tilde{V}_{,t}) - \frac{1}{2}D_\mu D_t^2 I_3(\rho u^2 - S_{\sigma\sigma}) \\
& -11I_1 Y_\mu - 2D_t^3 I_3(\rho u_\mu) - 140 D_\mu D_\sigma D_t^2 I_5[(\rho u_\sigma u_\nu - S_{\sigma\nu})_{,\nu} - \rho\tilde{V}_{,\sigma}] \\
& -\frac{3}{\pi}D_t I_1[3\tilde{V}_{,\mu}\tilde{V}_{,t} + 2(\tilde{V}_{,\sigma}\tilde{W}_{\sigma,\mu} - \tilde{V}_{,\mu\sigma}\tilde{W}_\sigma) + 2(\tilde{W}_\mu \Box \tilde{V} - \tilde{V}\Box\tilde{W}_\mu)]\Bigg\} \\
& +4\rho\tilde{V}_{,\sigma}D_t I_1\Bigg[\rho u_\mu u_\sigma - S_{\mu\sigma} - \frac{1}{4\pi}(\tilde{V}_{,\mu}\tilde{V}_{,\sigma} + 2\tilde{V}\tilde{V}_{,\mu\sigma}) \\
& +\frac{1}{3}\delta_{\mu\sigma}\left(\rho u^2 - S_{\nu\nu} + \frac{1}{2}\rho\tilde{V}\right)\Bigg]
\end{aligned} \tag{114}
$$

$$
\begin{aligned}
Z_1 = {} & -(\rho u^2 - S_{\sigma\sigma})\tilde{V}_{,t} - 4\tilde{W}_{\beta,\gamma}(\rho u_\beta u_\gamma - S_{\beta\gamma}) + \rho u_\beta D_\beta(4\tilde{K}_{\sigma\sigma} - \tilde{V}^2 - 2\tilde{N}) \\
& +\rho D_t(3\tilde{K}_{\sigma\sigma} - 2\tilde{N} - \tilde{V}^2 - \tfrac{1}{2}D_t^2 I_1\rho) - 4\rho\tilde{V}_{,\rho}\tilde{W}_\sigma
\end{aligned} \tag{115}
$$

$$
\begin{aligned}
Z_2 = {} & -\rho D_t I_1(\rho\tilde{V}_{,t}) + \tfrac{1}{3}\rho D_t^2 I_1(\rho u^2 - S_{\sigma\sigma} + \tfrac{1}{2}\rho\tilde{V}) \\
& -3\rho D_t^2 I_1(\rho u^2 - S_{\sigma\sigma}) + 2\rho D_t^2 I_1(\rho\tilde{V})
\end{aligned} \tag{116}
$$

$$Z_3 = -\frac{1}{2}(\rho u^2 - S_{\sigma\sigma}) D_t^2 I_2 \rho - 4(\rho u^2 - S_{\sigma\sigma}) \left(\tilde{V}_{,\nu} \tilde{W}_\nu + \tilde{V} \tilde{V}_{,t} + \frac{1}{4} \tilde{K}_{\sigma\sigma,t} \right)$$

$$+2(\rho u_\beta u_\gamma - S_{\beta\gamma}) \left[4(\tilde{W}_\beta \tilde{V}_{,\gamma} - \tilde{V} \tilde{W}_{\beta,\gamma}) - D_\gamma D_t^2 I_2(\rho u_\beta) \right.$$

$$\left. -\tilde{K}_{\beta\gamma,t} + \frac{1}{2\pi} \tilde{E}_{\beta\gamma,t} + \frac{1}{\pi} \tilde{F}_{\beta,\gamma} \right]$$

$$+\rho u_\beta [16(\tilde{W}_{\beta,\sigma} - \tilde{W}_{\sigma,\beta}) \tilde{W}_\sigma + 8 \tilde{V}_{,t} \tilde{W}_\beta]$$

$$+\rho u_\beta D_\beta \left[-\tilde{V} D_t^2 I_2 \rho + D_t^2 I_2(\rho u^2 - S_{\sigma\sigma}) - I_0 \rho D_t^2 I_2 \rho \right.$$

$$\left. -D_t^2 I_2(\rho \tilde{V}) - \frac{1}{\pi} \tilde{V} \tilde{E}_{\sigma\sigma} + 8 \tilde{W}_\sigma^2 - 4 \tilde{V} \tilde{K}_{\sigma\sigma} + 8 \tilde{V}^3 - \pi^{-1} \tilde{\beta} \right]$$

$$+\rho D_t \left[-\frac{1}{6} D_t I_3(\rho \tilde{V}_{,t}) - \frac{1}{24} D_t^4 I_4 \rho + \frac{3}{2} D_t^2 I_2(\rho u^2 - S_{\sigma\sigma}) \right.$$

$$\left. - I_0 \rho D_t^2 I_2 \rho - D_t^2 I_2(\rho \tilde{V}) - \pi^{-1} \mathscr{C} \right]$$

$$+\rho \tilde{V} D_t \left(-D_t^2 I_2 \rho + 10 \tilde{V}^2 - \frac{1}{\pi} \tilde{E}_{\sigma\sigma} - 4 \tilde{K}_{\sigma\sigma} \right)$$

$$+\rho \tilde{V}_{,t} \left(-D_t^2 I_2 \rho - 4 \tilde{K}_{\sigma\sigma} - \frac{1}{\pi} \tilde{E}_{\sigma\sigma} \right)$$

$$-2\rho \left[\tilde{V}_{,\sigma} D_t^2 I_2(\rho u_\sigma) + \tilde{W}_\sigma D_\sigma D_t^2 I_2 \rho + 4 \tilde{W}_\sigma \tilde{W}_{\sigma,t} \right.$$

$$\left. + \frac{1}{2\pi} \tilde{V}_{,\sigma} \tilde{F}_\sigma + 2 \tilde{W}_\sigma \tilde{K}_{\nu\nu,\sigma} - 4 \tilde{V} \tilde{V}_{,\sigma} \tilde{W}_\sigma \right] \tag{117}$$

$$Z_4 = -(\rho u^2 - S_{\sigma\sigma}) D_t I_1(\rho \tilde{V}_{,t}) + \frac{1}{3}(\rho u^2 - S_{\sigma\sigma}) D_t^2 I_1 \left[4(\rho u^2 - S_{\sigma\sigma}) + \frac{1}{2} \rho \tilde{V} \right]$$

$$+(\rho u_\beta u_\gamma - S_{\beta\gamma} \left[\frac{2}{3} D_\gamma D_t^3 I_3(\rho u_\beta) \right.$$

$$\left. +2 D_t^2 I_1(\rho u_\beta u_\gamma - S_{\beta\gamma}) - \frac{1}{2\pi} D_t^2 I_1(\tilde{V}_{,\beta} \tilde{V}_{,\gamma} + 2 \tilde{V} \tilde{V}_{,\beta\gamma}) \right]$$

$$
\begin{aligned}
&-2\rho u_\beta D_\beta\left[\frac{1}{3}D_t^3 I_3(\rho u^2-S_{\sigma\sigma})+I_0\rho I_1(\rho\tilde{V}_{,t})\right.\\
&\left.-\frac{1}{6}D_t^3 I_3(\rho\tilde{V})-\frac{1}{3}I_0 D_t I_1\left(\rho u^2-S_{\sigma\sigma}+\frac{1}{2}\rho\tilde{V}\right)\right]\\
&+2\rho u_\beta\tilde{V}_{,\beta}\left\{I_1\left[-\rho\tilde{V}_{,t}+\frac{1}{3}D_t\left(\rho u^2-S_{\sigma\sigma}+\frac{1}{2}\rho\tilde{V}\right)\right]\right.\\
&\left.+\frac{1}{2\pi}D_t I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})+2D_t I_1(\rho u^2-S_{\sigma\sigma})\right\}\\
&-2\rho\tilde{V}D_t\left[I_1(\rho\tilde{V}_{,t})-\frac{1}{3}D_t I_1(\rho u^2-S_{\sigma\sigma}+\frac{1}{2}\rho\tilde{V})\right.\\
&\left.-\frac{1}{2\pi}D_t I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})-2D_t I_1(\rho u^2-S_{\sigma\sigma})\right]\\
&+2\rho\tilde{V}_{,t}D_t\left\{\frac{1}{3}I_1\left[7(\rho u^2-S_{\sigma\sigma})+\frac{1}{2}\rho\tilde{V}\right]+\frac{1}{2\pi}I_1(\tilde{V}_{,\sigma}\tilde{V}_{,\sigma}+2\tilde{V}\tilde{V}_{,\sigma\sigma})\right\}\\
&+\rho D_t\left\{I_1 Z-\frac{1}{2}D_t^3 I_3(\rho u^2-S_{\sigma\sigma})-\frac{1}{3}D_t I_1[(x_\sigma-x'_\sigma)Y_\sigma]\right.\\
&+\frac{1}{120}D_t^3 I_5 D_\sigma[(\rho u_\sigma u_\nu)_{,\nu}-S_{\sigma\nu,\nu}-(\rho\tilde{V}_{,\sigma})]\\
&-2I_0\left[\rho I_1(\rho\tilde{V}_{,t})+\frac{1}{3}\rho D_t I_1\left(\rho u^2-S_{\sigma\sigma}+\frac{1}{2}\rho\tilde{V}\right)\right]\\
&\left.+D_t I_1\rho D_t^2 I_2\rho+\frac{1}{3}D_t^3 I_3(\rho\tilde{V})-\frac{1}{\pi}D_t I_1(\tilde{C})\right\}\\
&+\rho\tilde{V}_{,\sigma}\left\{4I_2 Y_\sigma+\frac{2}{3}D_t^3 I_3(\rho u_\sigma)+\frac{1}{\pi}D_t I_1[3\tilde{V}_{,\sigma}\tilde{V}_{,t}\right.\\
&\left.+2(\tilde{V}_{,\nu}\tilde{W}_{\nu,\sigma}-\tilde{V}_{,\nu\sigma}\tilde{W}_\nu)+2(\tilde{W}_\sigma\,\Box\,\tilde{V}-\tilde{V}\,\Box\,\tilde{W}_\sigma)]\right\}
\end{aligned}
\tag{118}
$$

As can be seen in (110)-(118), the decomposition used has the advantage that the equations of motion take an analogous form to the Newtonian ones. In fact, in the first three equations (110) may be seen the Newtonian equations of motion modified by the relativistic corrections Y_μ, Y'_μ, and Y''_μ, and in the fourth we see the equation of continuity with the terms Z_1, Z_2, Z_3, and Z_4, indicating change of mass. The first corrections go to $O(k^5)$ and the second to $O(k^{11/2})$. The Y_μ is of order $O(k^3)$, Y'_μ is $O(k^4)$, Y''_μ is

$O(k^{9/2})$, Z_1 is $O(k^{7/2})$, Z_2 is $O(k^4)$, Z_3 is $O(k^{9/2})$, and Z_4 is $O(k^5)$. The Y'' is the radiating term of the equation considered by Chandrasekhar and Esposito (1970) and by McCrea (1981) in connection with the quadrupole formula.

As Synge pointed out, there are advantages in this form of the equations of motion, such as the possibility of introducing additional bases of approximation in the physical situations in which $S_{\mu\nu}$ and u_μ are small. The concept of rigid motion and the junction conditions appear in a very natural form with these variables. If the motion is rigid, then u_μ must be a Killing vector satisfying

$$u_{\mu,\nu} + u_{\nu,\mu} = 0 \tag{119}$$

and the appropriate boundary conditions on the three-dimensional surface in which matter is confined is expressed by $T^{ab}n_b = 0$, where n_b is the covariant normal to that surface. Then, when a post-Newtonian formalism is used, it is easy to carry the results expressed with our variables to this formalism and see how, for instance, the approximate rigid conditions used in it may be derived in a natural way from (119). The fact that the only approximations introduced are in the right-hand side of equations (110) and (111) is, in short, the main advantage of the Eulerian formalism used. On one hand, as may be seen in (110) and (113) when the $O(k^4)$ and $O(k^{9/2})$ terms Y'_μ and Y''_μ are suppressed in (110) we have Synge's first three equations of motion in third approximation [cf. equations (1.61) and (1.62) of Synge (1970)]; on the other hand, in (111) and (115) and the first two terms of (116) we see Synge's fourth equation of motion [cf. (1.61) and (1.63) of Synge (1970)]. This is so because, although the term Z of Synge's fourth equation in $O(k^{7/2})$, two terms in it are $O(k^4)$ and naturally they appear in our approximation in the term Z_2, which is $O(k^4)$. So, with respect to Synge's third-order equations of motion, there are in (110) two more terms Y'_μ and Y''_μ that are $O(k^4)$ and $O(k^{9/2})$, respectively, and in (111) there appears in Z_2 all terms $O(k^4)$, and in Z_3 and Z_4 the $O(k^{9/2})$ and $O(k^5)$ corrections, respectively. Now, with the equations of motion in fourth approximation we are in a position to apply them to a great variety of problems, such as the study of orbits in concrete models, the two-body problem, and, in general, models corresponding to any situation of physical interest.

As an example let us consider the case of a perfect fluid with spherical symmetry and static. In order to apply these equations, first we must determine the value of the potentials appearing in them in terms of the functions that characterize the generating system of the field. Taking into account that this is static, the components of the metric tensor (49)-(52) are given by

$$g_{\alpha\beta} = [1 + 2(V + V^2 - K_{\sigma\sigma})]\delta_{\alpha\beta} + 4K_{\alpha\beta} + E_{\alpha\beta} + O(k^3) \tag{120}$$

$$g_{\alpha 4} = \qquad 0 \tag{121}$$

$$g_{44} = \qquad 1-2V+2V^2-2K_{\sigma\sigma}+4K_{\sigma\sigma}V-4V^3+\mathscr{A}+O(k^4) \tag{122}$$

because W_α and F_α are null in this case. The potentials (21), (23), and (24) are given by

$$V = -\int T^{44}(\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{123}$$

$$K_{\alpha\beta} = \int T^{\alpha\beta}(\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{124}$$

$$E_{\alpha\beta} = -\frac{1}{\pi}\int (V_{,\alpha}V_{,\beta}+2VV_{,\alpha\beta})\,|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{125}$$

and for (89) we have

$$\mathscr{A} = -\frac{1}{\pi}\int [-8\pi T^{44}K_{\sigma\sigma}-8\pi VT^{\sigma\sigma}+2V_{,\sigma\nu}K_{\sigma\nu}+\tfrac{1}{2}V_{,\sigma\nu}E_{\sigma\nu}$$
$$-2VK_{\sigma\nu,\sigma\nu}-\tfrac{1}{2}VE_{\sigma\nu,\sigma\nu}+VV_{,\sigma}^2+8T^{\sigma\sigma}V^2]\,|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{126}$$

Clearly, whereas the density functions in (123) and (124) are of compact support (and so their respective integrals are extended over the domain c occupied by the material system), the density functions in (125) and (126) are nonnull all over the space (in fact, when the distance r increases, they decrease like r^{-4}) and so their corresponding integrals are also extended all over the space.

Since in the system we are working with, $G = c = 1$, we have for V the value

$$V = M/r \tag{127}$$

where $r = |\mathbf{x}|$ and M is the mass of the material system given by

$$M = \int_c T^{44}\,d_3x' \tag{128}$$

In order to determine $K_{\alpha\beta}$, let us denote by $p(x')$ the distribution of pressure inside the material system. We then have

$$T^{\alpha\beta} = p(x')\delta_{\alpha\beta} \tag{129}$$

and

$$K_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{r}\int_c p(x')\,d_3x' \tag{130}$$

Now, for the calculus of $p(x')$ in the order of approximation we are considering we have to solve the equilibrium equation

$$p_{,\alpha} = \rho V_{,\alpha} + T^{\sigma\sigma} V_{,\alpha} - 4T^{\alpha\beta} V_{,\beta} - 4\rho V V_{,\alpha} + \rho K_{\sigma\sigma,\alpha} + O(k^4) \tag{131}$$

which in turn requires previous knowledge of $K_{\sigma\sigma}$ inside the material system and in the former approximation. Since the hydrostatic equilibrium equation in this approximation is given in polar coordinates by

$$\frac{dp(r)}{dr} = \rho \frac{dV}{dr} + O(k^3) \tag{132}$$

then, using the value of V inside the material system, which is given by

$$V = \tfrac{2}{3}\pi T^{44}(3R^2 - r^2) \tag{133}$$

(where R denotes the system's radius) and the boundary condition $p = 0$ at $r = R$, from (132) and (133) the following pressure distribution is obtained:

$$p(r) = \tfrac{2}{3}\pi (T^{44})^2 (R^2 - r^2) \tag{134}$$

So, for the former approximation

$$K_{\sigma\sigma} = -\frac{4\pi}{r}(T^{44})^2 \int_0^R dr' \, (r'^3 - Rr')[r^2 + r'^1 - 2rr' \cos\theta]^{1/2}\Big|_0^\pi + O(k^3) \tag{135}$$

Now, if we take into account that

$$[r^2 + r'^2 - 2rr' \cos\theta]^{1/2} = \begin{cases} 2r' & \text{if} \quad r > r' \\ 2r & \text{if} \quad r < r' \end{cases} \tag{136}$$

then from (135) we have

$$K_{\sigma\sigma} = -4\pi^2 (T^{44})^2 \left[-\frac{1}{10} r^4 + \frac{1}{3} Rr^2 + \left(\frac{R^4}{2} - R^3 \right) \right] + O(k^3) \tag{137}$$

Carrying then (137) to (131), we finally have the following differential equation:

$$\frac{dp}{dr} + arp(r) = br + cr^3 + O(k^4) \tag{138}$$

where

$$a = -\tfrac{4}{3}\pi T^{44} rp(r) = O(k^3) \tag{139}$$

$$b = -\tfrac{4}{3}\pi (T^{44})^2 + \tfrac{32}{3}\pi^2 (T^{44})^3 R^2 - \tfrac{8}{3}\pi^2 R (T^{44})^3 = O(k^2) \tag{140}$$

$$c = -\tfrac{88}{45}\pi^2 (T^{44})^3 = O(k^3) \tag{141}$$

The solution of (138) [with, as before, $p(r) = 0$ for $r = R$] is

$$p(r) = \left[\frac{b}{a} - \frac{2c}{a} + \frac{c}{a} R^2 \right] \exp\left[(R^2 - r^2) \frac{a}{2} \right] + \left[\frac{b}{a} - \frac{2c}{a^2} + \frac{c}{a} r^2 \right] + O(k^4) \tag{142}$$

Then, substituting (142) in (130) after a McLaurin series expansion of the exponential function in (142), we finally have the value of $K_{\alpha\beta}$ in the desired approximation:

$$K_{\alpha\beta}=\frac{\delta_{\alpha\beta}}{r}\left[\frac{\pi}{5R}M^2+\frac{3\pi}{5R^2}\left(\frac{2}{5R}-\frac{10}{7}\right)M^3\right]+O(k^4) \tag{143}$$

Now, in accord with what has been previously said, the value of $E_{\alpha\beta}$ must also be obtained. $E_{\alpha\beta}$ must be a solution of the differential equation

$$\Delta E_{\alpha\beta}=4M^2\left(\frac{7x_\alpha x_\beta}{r^6}-2\frac{\delta_{\alpha\beta}}{r^4}\right) \tag{144}$$

which, due to the symmetry of the model we are considering, must be of the form

$$E_{\alpha\beta}=-\delta_{\alpha\beta}f_1(r)+(x_\alpha x_\beta-\tfrac{1}{3}\delta_{\alpha\beta}r^2)f_2(r) \tag{145}$$

Then, taking the Laplacian of (145), from (144) and (145) we obtain the equations

$$\frac{d^2f_1(r)}{dr^2}+\frac{2}{r}\frac{df_1(r)}{dr}=-\frac{4}{3}(V')^2 \tag{146}$$

$$\frac{d^2f_2(r)}{dr^2}+\frac{6}{r}\frac{df_2(r)}{dr}=\frac{28}{r}(V')^2 \tag{147}$$

Using then the boundary conditions

$$\lim_{r\to\infty}f_i(r)=0 \qquad (i=1,2) \tag{148}$$

we obtain the following as the solution of these differential equations

$$f_1(r)=-\frac{2}{3}\frac{M^2}{r^2}+\frac{8}{3}\frac{\mathscr{E}}{r} \tag{149}$$

$$f_2=-\frac{21}{5}\frac{M^2}{r^4}+\frac{28}{5}\frac{M^2}{r^5}L \tag{150}$$

where $\mathscr{E}$ and L are constants given by

$$\mathscr{E}=-3M^2/5R \tag{151}$$

$$L=\tfrac{6}{7}R \tag{152}$$

Carrying then (149) and (150) to (145), we finally have for $E_{\alpha\beta}$ the value

$$E_{\alpha\beta}=\left(\frac{31}{15}\frac{M^2}{r^2}-\frac{8}{3}\frac{\mathscr{E}}{r}-\frac{28}{15}\frac{M^2}{r^3}L\right)\delta_{\alpha\beta}-\left(\frac{21}{5}\frac{M^2}{r^4}-\frac{28}{5}\frac{M^2}{r^5}L\right)x_\alpha x_\beta \tag{153}$$

Finally, in order to obtain the whole expression for the gravitational field we are considering, we need the value of $\mathscr{A}$. In the previous calculations we have only used symmetries and the weakness of the field. But now we know that if $f(\mathbf{x})$ is a function that decreases like r^{-4} when $r \to \infty$, then

$$\int f(\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' = \frac{1}{r}\int f(\mathbf{x}')\,d_3x' + O(r^{-2}\log r) \tag{154}$$

the integral being extended all over the space. So, maintaining the principal part for large values of r, from (126) we have

$$\mathscr{A} = \frac{1}{r}\left\{8\int(-T^{44}K_{\sigma\sigma} - VT^{\sigma\sigma})\,d_3x' + \int(\pi^{-1}VV^2_{,\sigma} + 8T^{44}V^2)\,d_3x'\right.$$
$$\left. + (2\pi)^{-1}\int(\Sigma_{\sigma\nu}V_{,\sigma\nu} - V\Sigma_{\sigma\nu,\sigma\nu})\,d_3x'\right\} + O(k^3r^{-2}\log r) \tag{155}$$

where

$$\Sigma_{\alpha\beta} = 4K_{\alpha\beta} + E_{\alpha\beta} \tag{156}$$

The first integral in (155) is equivalent to

$$8\iint d_3x'\,d_3x''\,[T^{44}(x')T^{\sigma\sigma}(x'') - T^{\sigma\sigma}(x')T^{44}(x'')]\,|\mathbf{x}-\mathbf{x}'|^{-1} \tag{157}$$

which, due to the spherical symmetry, is null because it is extended over all the space.

Using the integration by parts, we obtain from the second integral

$$10\int T^{44}(x')V^2(x')\,d_3x' \tag{158}$$

that is,

$$\frac{54}{9}\frac{M^3}{R^2} \tag{159}$$

and finally applying again the integration by parts and the condition that $\Sigma_{\mu\nu}$ and its first derivatives are null at infinity, we see that the last integral in (155) is also null over all the space. So, far from the generating system of the field, $\mathscr{A}$ is given by

$$\mathscr{A} = -\frac{59}{4}\frac{M^3}{R^2}\frac{1}{r} + O(r^{-2}\log r) \tag{160}$$

The potentials (127), (143), (153), and (160) determined the metric tensor (120)–(123) of the space-time in the order desired. As may be seen, these potentials are given only in terms of the mass of the generating system of the field. The potential $\Sigma_{\alpha\beta}$ is given by

$$\Sigma_{\alpha\beta} = \phi_1\delta_{\alpha\beta} + \phi_2 x_\alpha x_\beta \tag{161}$$

where

$$\phi_1 = \frac{4\pi}{5R}\frac{M^2}{r} + \frac{31}{15}\frac{M^2}{r^2} - \frac{8}{3}\frac{\mathscr{E}}{r} - \frac{28}{15}\frac{M^2}{r^3}L + O(k^3) \tag{162}$$

$$\phi_2 = -\frac{21}{5}\frac{M^2}{r^4} + \frac{28}{5}\frac{M^2}{r^5}L + O(k^3) \tag{163}$$

Now, carrying (127), (143), (153), and (160) to equations (110) and (111) we can obtain the equations of motion for a particle moving in this field. To do this we only need to add the hypothesis that the particle does not modify the potentials already obtained. Neglecting, then, the self-potentials of the particle and taking into account the static character of the field, from (110) and (112)-(114) we obtain the simpler equations

$$\begin{aligned} \dot{u}_\rho = {} & V_{,\rho} - 4V_{,\mu}u_\mu u_\rho + V_{,\rho}u^2 - 2(V^2)_{,\rho} + K_{\sigma\sigma,\rho} \\ & + 4(V^3)_{,\rho} - K_{\sigma\sigma,\rho}u^2 - 4K_{\mu\rho,\nu}u_\mu u_\nu - \frac{1}{\pi}E_{\rho\alpha}V_{,\alpha} \\ & + 2K_{\mu\nu,\rho}u_\mu u_\nu - \frac{1}{2}\left(\frac{2}{\pi}E_{\mu\rho,\nu} - \frac{1}{\pi}E_{\mu\nu,\rho}\right)u_\mu u_\nu \\ & - 4VK_{\sigma\sigma,\rho} - 4K_{\rho\alpha}V_{,\alpha} - \mathscr{A}_{,\rho} + O(k^4) \end{aligned} \tag{164}$$

with V, $K_{\alpha\beta}$, $E_{\alpha\beta}$, and $\mathscr{A}$ as in (127), (143), (153), and (160).

These equations have obviously two first integrals, from which we can obtain the trajectory of any particle. In fact, since the field is static, we have the integral of energy

$$\begin{aligned} \tfrac{1}{2}u^2 = {} & E + V - 5V^2 + K_{\sigma\sigma} - 6EV + 14V^3 + 9VE^2 - 6VK_{\sigma\sigma} + 18EV^2 \\ & - 2EK_{\sigma\sigma} - \tfrac{3}{2}E^2 + 2E^3 - \tfrac{1}{2}(E_{\mu\nu} + 4K_{\mu\nu})u_\mu u_\nu - \tfrac{1}{2}\mathscr{A} + O(k^4) \end{aligned} \tag{165}$$

and, due to the spherical symmetry, we have the integral of angular momentum

$$r^2\dot{\phi} = h[1 - 4V + 8V^2 + E^2 - \phi_1] + O(k^{7/2}) \tag{166}$$

(their derivation can be found in Appendix C.)

Combining then (165) and (166) in the usual way (see, e.g., Gambi, 1985), we obtain for the trajectories the following equation:

$$\begin{aligned} \left(\frac{d\xi}{d\theta}\right)^2 = {} & 2h^{-2}\left(E - \frac{3}{2}E^2\right) + 2h^{-2}\left[M + 2EM + \frac{3\pi}{5R}M^2 + \frac{9}{2R^2}\left(\frac{1}{5R} - \frac{5}{7}\right)M^3\right. \\ & \left. - \frac{2\pi}{5R}EM^2 + 7E^2M - \frac{8}{3}E\mathscr{E} + \frac{59}{8R^2}M^3\right]\xi \\ & + \left[-1 + 2h^{-2}\left(3M^2 + \frac{124}{15}EM^2 - \frac{8}{3}M\mathscr{E} + \frac{2\pi}{R}M^3\right)\right]\xi^2 \end{aligned}$$

$$+\left[-\frac{8}{3}\mathscr{E}+\frac{4\pi}{5R}M^2+2h^{-2}\left(\frac{184}{15}M^3-\frac{56}{15}M^2EL\right)\right]\xi^3$$

$$+\left[-\frac{32}{15}M^2-\frac{112}{15}h^{-2}M^3L\right]\xi^4$$

$$+\frac{56}{15}LM^2\xi^5+O(k^3r^{-2}\log r); \qquad r=\frac{1}{\xi} \tag{167}$$

In it we can see the classical term

$$\xi^{-2}-2h^{-2}(E+V) \tag{168}$$

and the terms corresponding to the fomer approximation

$$-2h^{-2}(2K_{\sigma\sigma}+6V^2+6EV-3E^2) \tag{169}$$

In order to see the behavior of these trajectories, let us calculate the advance of perihelion of any bounded one. If we denote the apsidal points by ξ_1 and ξ_2 (with $0<\xi_1<\xi_2$), we know that the apsidal angle is given by

$$\Delta\theta=\int_{\xi_1}^{\xi_2}[-F(\xi)]^{-1/2}\,d\xi \tag{170}$$

where $F(\xi)$ is given by the right-hand side of (167).

In order to avoid the singularities in the integrand, we write $F(\xi)$ in the form

$$F(\xi)=\xi^2-2Mh^{-2}\xi-2Eh^{-2}+\sum_{\alpha=0}^{5}a_\alpha\xi^\alpha+O(k^3r^{-2}\log r) \tag{171}$$

where a_α is the coefficient of ξ^α, having removed the first three terms in (171). Clearly a_0, a_1, and a_2 are $O(k)$, and a_3, a_4, and a_5 are $O(k^2)$. Writing now $F(\xi)$ in the form

$$F(\xi)=[\xi^2-(\xi_1+\xi_2)\xi+\xi_1\xi_2][1+G(\xi)] \tag{172}$$

where

$$G(\xi)=\frac{(-2Mh^{-2}+\xi_1+\xi_2)\xi-(\xi_1\xi_2+2Eh^{-2})+\sum_{\alpha=0}^{5}a_\alpha\xi^\alpha}{(\xi-\xi_1)(\xi-\xi_2)} \tag{173}$$

and taking into account that $F(\xi_1)=F(\xi_2)=0$, we have

$$-2Mh^{-2}+\xi_1+\xi_2=\frac{\xi_1\sum_{\alpha=0}^{5}a_\alpha\xi_2^\alpha-\xi_2\sum_{\alpha=0}^{5}a_\alpha\xi_1^\alpha}{\xi_1-\xi_2} \tag{174}$$

$$2Eh^{-2}+\xi_1\xi_2=\frac{\xi_1\sum_{\alpha=0}^{5}a_\alpha\xi_2^\alpha-\xi_2\sum_{\alpha=0}^{5}a_\alpha\xi_1^\alpha}{\xi_1-\xi_2} \tag{175}$$

from which we deduce

$$G(\xi)=\sum_{\alpha=0}^{5}a_\alpha\frac{(\xi-\xi_1)(\xi_2^\alpha-\xi^\alpha)+(\xi-\xi_2)(\xi^\alpha-\xi_1^\alpha)}{(\xi_1-\xi_2)(\xi-\xi_1)(\xi-\xi_2)} \tag{176}$$

Using now the binomial theorem, we have from (170) and (176)

$$\Delta\theta=\int_{\xi_1}^{\xi_2}\frac{1}{[(\xi_1-\xi)(\xi-\xi_2)]^{1/2}}\left[1-\frac{1}{2}G(\xi)+\frac{3}{8}G^2(\xi)\right]d\xi+O(k^3) \tag{177}$$

i.e.,

$$\begin{aligned}\Delta\theta=S_1&+h^{-2}\left(3M^2+\frac{124}{15}EM^2-\frac{8}{3}M\mathscr{E}+\frac{2\pi}{R}M^3\right)S_2\\&+\left[-\frac{4}{3}\mathscr{E}+\frac{2\pi}{5R}M^2+h^{-2}\left(\frac{184}{15}M^3-\frac{56}{5}M^2EL\right)\right]S_3\\&-\left(\frac{16}{15}M^2+\frac{56}{15}h^{-2}M^3L\right)S_4\\&+\frac{28}{15}LM^2S_5+\frac{27}{2}(h^{-2}M^2)^2S_6+O(k^3r^{-2}\log r)\end{aligned} \tag{178}$$

where

$$S_1=\int_{\xi_1}^{\xi_2}\frac{1}{[(\xi_1-\xi)(\xi-\xi_2)]^{1/2}}d\xi \tag{179}$$

$$S_j=\int_{\xi_1}^{\xi_2}\frac{(\xi-\xi_1)(\xi_2^j-\xi^j)+(\xi-\xi_2)(\xi^j-\xi_1^j)}{(\xi_1-\xi_2)(\xi-\xi_1)(\xi-\xi_2)[(\xi_1-\xi)(\xi-\xi_2)]^{1/2}}d\xi \qquad (j=2,\ldots,5) \tag{180}$$

$$S_6=\int_{\xi_1}^{\xi_2}\frac{[(\xi-\xi_1)(\xi_2^2-\xi^2)+(\xi-\xi_2)(\xi^2-\xi_1^2)]^2}{[(\xi_1-\xi_2)(\xi-\xi_1)(\xi-\xi_2)]^2[(\xi_1-\xi)(\xi-\xi_2)]^{1/2}}d\xi \tag{181}$$

Taking into account that

$$-2Mh^{-2}+\xi_1+\xi_2=O(k) \tag{182}$$

$$-2Eh^{-2}-\xi_1\xi_2=O(k) \tag{183}$$

a direct calculation of (179)-(181) gives

$$S_1 = S_2 = S_6 = \pi \tag{184}$$

$$S_3 = 3\pi M h^{-2} + O(k) \tag{185}$$

$$S_4 = \pi(\tfrac{9}{2}M^2 h^{-4} + 3Eh^{-2}) + O(k) \tag{186}$$

$$S_5 = \pi(5M^3 h^{-6} - \tfrac{5}{4}M^2 h^{-4} + \tfrac{15}{2}EMh^{-2} - \tfrac{17}{12}Mh^{-2}) + O(k) \tag{187}$$

So, the advance of perihelion is

$$\begin{aligned}\theta = {} & 6\pi M^2 h^{-2} + 8\pi E M^2 h^{-2} + \frac{8}{3}\pi \mathscr{E} M h^{-2} + \frac{32}{5}\frac{\pi^2}{R} M^3 h^{-2} \\ & + \left(91 - \frac{14}{3}L\right)\pi M^4 h^{-4} - \frac{364}{15}\pi L E M^3 h^{-4} - \frac{238}{45}\pi L M^3 h^{-2} \\ & + \frac{224}{15}\pi L M^5 h^{-6} + O(k^3 r^{-2}\log r)\end{aligned} \tag{188}$$

As can be seen, it includes all the relativistic corrections, such as the Schwarzschild advance, that correspond to the former order of approximation.

APPENDIX A

Equations (13) may be written in the form

$$T^{ab}{}_{,b} = X_a \tag{A1}$$

where

$$X_a = -(\Gamma^a_{bc} + \tfrac{1}{2}\delta_{ab}\Gamma^d_{cd} + \tfrac{1}{2}\delta_{ac}\Gamma^d_{bd})T^{bc} + O(k^5) \tag{A2}$$

Γ^a_{bc} being the Christoffel symbol of the second kind with respect to the metric (35), i.e.,

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ah}(g_{bh,c} + g_{ch,b} - g_{bc,h}) \tag{A3}$$

where

$$g_{ab} = \delta_{ab} + p_{ab} + q_{ab} + r_{ab} + O(k^4) \tag{A4}$$

with

$$p_{ab} = O(k), \qquad q_{ab} = O(k^2), \qquad r_{ab} = O(k^3) \tag{A5}$$

With the order of approximation considered we have

$$g^{ab} = \delta_{ab} - p_{ab} - q_{ab} - r_{ab} + p_{al}p_{lb} + 2p_{al}q_{lb} - p_{al}p_{lm}p_{mb} + O(k^4) \tag{A6}$$

Carrying then (A4) and (A6) to (A3), we have

$$\Gamma^a_{bc} = \tfrac{1}{2}(p_{ab,c} + q_{ab,c} + r_{ab,c} + p_{ac,b} + q_{ac,b} + r_{ac,b} - p_{bc,a} - q_{bc,a} - r_{bc,a}$$
$$- p_{ah}p_{bh,c} - p_{ah}q_{bh,c} - p_{ah}p_{ch,b} - p_{ah}q_{ch,b} + p_{ah}p_{bc,h} + p_{ah}q_{bc,h}$$
$$- q_{ha}p_{bh,c} - q_{ah}p_{ch,b} + q_{ah}p_{bc,h} + p_{al}p_{lh}p_{bh,c} + p_{al}p_{lh}p_{ch,b} - p_{al}p_{lh}p_{bc,h}) \quad ((A7)$$

Then, taking into account (A2), we have

$$X_a = -\tfrac{1}{2}p_{dd,c}T^{ac} + (\tfrac{1}{2}p_{bc,a} - p_{ab,c})T^{bc} + (\tfrac{1}{4}p_{mn}p_{mn} - \tfrac{1}{2}q_{mn})_{,c}T^{ac}$$
$$+ (p_{an}p_{bn,c} - \tfrac{1}{2}p_{an}p_{bc,n} + \tfrac{1}{2}q_{bc,a} - q_{ab,c})T^{bc}$$
$$-\tfrac{1}{2}r_{nn,c}T^{ac} + \tfrac{1}{2}(p_{dm}q_{dm})_{,c}T^{ac} - \tfrac{1}{2}p_{dl}p_{lh}p_{dh,c}T^{ac} - (r_{ab,c} - \tfrac{1}{2}r_{bc,a})T^{bc}$$
$$+ p_{am}q_{bm,c}T^{bc} - \tfrac{1}{2}(p_{ah}q_{bc,h} + q_{ah}p_{bc,h})T^{bc}$$
$$+ q_{ah}p_{bh,c}T^{bc} - p_{am}p_{mh}p_{bh,c}T^{bc} + \tfrac{1}{2}p_{am}p_{mn}p_{bc,n}T^{bc} + O(k^5) \quad (A8)$$

and from here we have (38)-(40).

APPENDIX B

Initially the expansion for V is

$$V = \tilde{V} + D_t I_1 T^{44} - \frac{1}{2} D_t^2 I_2 T^{44} + \frac{1}{3!} D_t^3 I_3 T^{44}$$
$$- \frac{1}{4!} D_t^4 I_4 T^{44} + \frac{1}{5!} D_t^5 I_5 T^{44} + O(k^4) \quad (B1)$$

the integrals being extended all over space. Now since the density functions are supposed to be of compact support, these integrals may be extended only over the region v occupied by matter. Using Synge's third-order equations of motion

$$\rho \dot{u}_\mu + u_\mu(\dot{\rho} + \rho\theta) - S_{\mu\nu,\nu} = \rho \tilde{V}_{,\mu} + Y_\mu + O(k^4) \quad (B2)$$
$$\dot{\rho} + \rho\theta = -\rho \tilde{V}_{,t} + Z + O(k^{9/2}) \quad (B3)$$

with definitions (105)-(107), where Y_μ and Z are given by (64) and (65), respectively, we can evaluate these integrals in the following way:

For $D_t I_1 T^{44}$ we have

$$D_t I_1 T^{44} = \int_v T^{44}{}_{,t}\, d_3x'$$
$$= \int_v [\dot{T}^{44} - i(T^{44}{}_{,\mu}T^{\mu 4})/T^{44}]\, d_3x'$$
$$= \int_v [i(T^{44}T^{\mu 4}/T^{44})_{,\mu} + T^{44}\tilde{V}_{,t} + Z]\, d_3x' + O(k^{9/2})$$

$$= -\int_v T^{44}\tilde{V}_{,t}\, d_3x' - \int_v Z\, d_3x' + O(k^{9/2})$$

$$= -I_1(T^{44}\tilde{V}_{,t} + Z) + O(k^{9/2}) \tag{B4}$$

On the other hand

$$\frac{1}{6} D_t^3 I_3 T^{44} = -\frac{1}{6} D_t \int [i(T^{44}T^{\mu 4}/T^{44})_{,\mu t} + (T^{44}\tilde{V}_{,t})_{,t} + O(k^4)]|\mathbf{x}-\mathbf{x}'|^2\, d_3x'$$

$$= -\frac{1}{6} D_t \int [(T^{\mu\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\mu} - Y_\mu)_{,\mu} + (T^{44}\tilde{V}_{,t})_{,t}$$

$$+ O(k^4)]|\mathbf{x}-\mathbf{x}'|^2\, d_3x'$$

$$= -\frac{1}{6} D_t \int (T^{\mu\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\mu})_{,\mu} |\mathbf{x}-\mathbf{x}'|^2\, d_3x'$$

$$+ \frac{1}{6} D_t \int Y_{\mu,\mu} |\mathbf{x}-\mathbf{x}'|^2\, d_3x' + \frac{1}{6} D_t^2 I_3(\rho\tilde{V}_{,t}) + O(k^4) \tag{B5}$$

But

$$\int_v (T^{\mu\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\mu})_{,\mu} |\mathbf{x}-\mathbf{x}'|^2\, d_3x'$$

$$= 2\left[\int_v T^{\sigma\sigma}\, d_3x' + \int_v x_\sigma T^{44}\tilde{V}_{,\sigma}\, d_3x' - \int_v x'_\sigma T^{44}\tilde{V}_{,\sigma}\, d_3x'\right]$$

$$= 2\int_v T^{\sigma\sigma}\, d_3x' - 2\int_v x'_\sigma T^{44}\tilde{V}_{,\sigma}\, d_3x' \tag{B6}$$

because

$$\int_v T^{44}\tilde{V}_{,\sigma}\, d_3x' = \int_v T^{44}(T^{44})' \frac{(x_\sigma - x'_\sigma)}{|\mathbf{x}-\mathbf{x}'|^3}\, d_3x\, d_3x' = 0 \tag{B7}$$

Now

$$-\int_v x'_\sigma T^{44}\tilde{V}_{,\sigma}\, d_3x'$$

$$= \int_v x'_\sigma T^{44}\left[\int_v (T^{44})' \frac{x_\sigma - x'_\sigma}{|\mathbf{x}-\mathbf{x}'|^3}\, d_3x\right] d_3x'$$

$$= \int_v \int_v T^{44}(T^{44})' x'_\sigma \frac{x_\sigma - x'_\sigma}{|\mathbf{x}-\mathbf{x}'|^3}\, d_3x\, d_3x'$$

$$= \int_v \int_v T^{44}(T^{44})' x_\sigma \frac{x'_\sigma - x_\sigma}{|\mathbf{x}-\mathbf{x}'|^3}\, d_3x\, d_3x'$$

$$=\frac{1}{2}\int_v\int_v T^{44}(T^{44})'\frac{(x'_\sigma-x_\sigma)(x_\sigma-x'_\sigma)}{|\mathbf{x}-\mathbf{x}'|^3}\,d_3x\,d_3x'$$

$$=-\frac{1}{2}\int_v\int_v T^{44}(T^{44})'|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x\,d_3x'$$

$$=-\frac{1}{2}\int_v T^{44}\tilde{V}\,d_3x' \tag{B8}$$

then

$$\int_v (T^{\mu\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\mu})_{,\mu}|\mathbf{x}-\mathbf{x}'|^3\,d_3x'=2\int_v T^{\sigma\sigma}\,d_3x'-\int_v T^{44}\tilde{V}\,d_3x' \tag{B9}$$

On the other hand, since

$$\int_v Y_{\mu,\mu}|\mathbf{x}-\mathbf{x}'|^2\,d_3x'=2\int_v (x_\mu-x'_\mu)\,Y_\mu\,d_3x' \tag{B10}$$

then, taking into accout (B5), (B9), and (B10), we have

$$\frac{1}{6}D_t^3I_3T^{44}=\frac{1}{3}D_tI_1(T^{\sigma\sigma}-\frac{1}{2}T^{44}\tilde{V}) + \frac{1}{3}D_t\int (x_\mu-x'_\mu)\,Y_\mu\,d_3x' - \frac{1}{6}D_t^2I_3(T^{44}\tilde{V}_{,t})+O(k^4) \tag{B11}$$

On the other hand, since

$$\frac{1}{5!}D_t^5I_5T^{44}=-\frac{1}{5!}D_t^3\int (T^{\mu\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\mu})_{,\mu}|\mathbf{x}-\mathbf{x}'|^4\,d_3x'+O(k^4) \tag{B12}$$

then, taking into account (B4), (B11), and (B12), we have finally

$$V=\tilde{V}-I_1(T^{44}\tilde{V}_{,t}+Z)-\frac{1}{2}D_t^2I_2T^{44}-\frac{1}{3}D_tI_1\left(T^{\sigma\sigma}-\frac{1}{2}T^{44}V\right) + \frac{1}{3}D_t\int (x_\mu-x'_\mu)\,Y_\mu\,d_3x'-\frac{1}{6}D_t^2I_3(T^{44}\tilde{V}_{,t}) - \frac{1}{4!}D_t^4I_4T^{44}-\frac{1}{5!}D_t^3I_5(T^{\mu\nu}{}_{,\nu}+T^{44}\tilde{V}_{,\mu})_{,\mu}+O(k^4) \tag{B13}$$

that is to say, (63).

For the potential W_μ we initially have

$$W_\mu = \tilde{W}_\mu + iD_t I_1(T^{4\mu}) - \tfrac{1}{2} i D_t^2 I_2(T^{4\mu}) + \tfrac{1}{6} i D_t^3 I_3(T^{4\mu}) + O(k^{7/2}) \tag{B14}$$

but, since from (B2) we have

$$i(T^{4\mu})_{,t} = T^{\mu\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\mu} - Y_\mu + O(k^4) \tag{B15}$$

then

$$\begin{aligned} iD_t I_1(T^{4\mu}) &= i\int (T^{4\mu})_{,t}\, d_3x' \\ &= \int (T^{\mu\nu}{}_{,\nu} + T^{44}\tilde{V}_{,\mu} - Y_\mu)\, d_3x' + O(k^4) \\ &= -\int Y_\mu\, d_3x' + O(k^4) \end{aligned} \tag{B16}$$

with which (B14) becomes

$$W_\mu = \tilde{W}_\mu - I_1(Y_\mu) - \tfrac{1}{2} D_t^2 I_2(iT^{4\mu}) + \tfrac{1}{6} D_t^3 I_3(iT^{4\mu}) + O(k^{7/2}) \tag{B17}$$

that is to say (69).

Now with V and W_μ given by (B13) and (B17), respectively, the remaining expansions from (67) to (95) are easily derived.

APPENDIX C

Since the gravitational field we are considering is static, the Lagrangian $\mathscr{L}$ does not depend explicitly on time and so the following quantity is conserved:

$$\mathscr{L} - u_\mu\, \partial\mathscr{L}/\partial u_\mu \tag{C1}$$

Then, we can write

$$\mathscr{L}^{-1}\left(\mathscr{L}^2 - \frac{1}{2} u_\mu\, \partial\mathscr{L}/\partial u_\mu\right) - 1 = E \tag{C2}$$

where, taking into account (14) and (102)-(104),

$$\mathscr{L} = [1 + \gamma_{44} - (\delta_{\mu\nu} + \gamma_{\mu\nu}) u_\mu u_\nu + O(k^4)]^{1/2} \tag{C3}$$

Now, expanding $\mathscr{L}^{-1}$, we have

$$\begin{aligned} \mathscr{L}^{-1} = 1 &- \tfrac{1}{2}(\gamma_{44} - u^2 - \gamma_{\mu\nu} u_\mu u_\nu) \\ &+ \tfrac{3}{8}(\gamma_{44}^2 - 2\gamma_{44} u^2 - 2\gamma_{\mu\nu}\gamma_{44} u_\mu u_\nu + u^4 + 2\gamma_{\mu\nu} u_\mu u_\nu u^2) \\ &- \tfrac{5}{16}(\gamma_{44}^3 - 3\gamma_{44}^2 u^2 + 3\gamma_{44} u^4 - u^6) + O(k^4) \end{aligned} \tag{C4}$$

and then, carrying (C3) and (C4) to (C2) and taking into account (120)-(122), we have

$$E=\tfrac{1}{2}u^2-V+\tfrac{1}{8}V^2-K_{\sigma\sigma}+\tfrac{3}{2}Vu^2-\tfrac{1}{2}u^2K_{\sigma\sigma}$$
$$+VK_{\sigma\sigma}+\tfrac{3}{8}u^4+\tfrac{9}{4}V^2u^2-\tfrac{3}{2}V^3+\tfrac{21}{8}Vu^4$$
$$+\tfrac{1}{2}(E_{\mu\nu}+4K_{\mu\nu})u_\mu u_\nu+\tfrac{5}{16}u^6+\tfrac{1}{2}\mathscr{A}+O(k^4) \tag{C5}$$

Finally, if we take into account that

$$u^2=2E+2V-10V^2-3E^2+2K_{\sigma\sigma}-12EV+O(k^3) \tag{C6}$$

then, from (C5) and (C6) we have

$$\tfrac{1}{2}u^2=E+V-5V^2+K_{\sigma\sigma}-6EV+14V^3+9VE^2$$
$$-6VK_{\sigma\sigma}+18EV^2-2EK_{\sigma\sigma}-\tfrac{3}{2}E^2+2E^3$$
$$-\tfrac{1}{2}(E_{\mu\nu}+4K_{\mu\nu})u_\mu u_\nu-\tfrac{1}{2}\mathscr{A}+O(k^4) \tag{C7}$$

that is to say, (165).

On the other hand, due to the spherical symmetry, the following quantity is also conserved:

$$x_1\,\partial\mathscr{L}/\partial\dot{x}_2-x_2\,\partial\mathscr{L}/\partial\dot{x}_1 \tag{C8}$$

We can then write

$$\mathscr{L}^{-1}[x_1g_{2\mu}\dot{x}_\mu-x_2g_{\mu 1}\dot{x}_\mu+i(x_1g_{24}-x_2g_{14})]=A \tag{C9}$$

where, as before, $\mathscr{L}^{-1}$ is given by (C4). Then, using (102)-(104), (120)-(122), and (161), from (C9) we have in polar coordinates

$$A=r^2\dot{\phi}\left[1+3V+\frac{9}{2}V^2-K_{\sigma\sigma}+\frac{u^2}{2}+\frac{7}{2}Vu^2+\frac{3}{8}u^4+\phi_1\right]+O(k^{7/2}) \tag{C10}$$

with ϕ_1 as in (162).

Now, since

$$AE=r^2\dot{\phi}[1+4V+E]E+O(k^{7/2}) \tag{C11}$$

defining h by

$$h=A(1-E) \tag{C12}$$

from (C10) and (C11) we have

$$r^2\dot{\phi}=h[1-4V+8V^2+E^2-\phi_1]+O(k^{7/2}) \tag{C13}$$

that is to say, (166).

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